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Small-world behaviour in a system of mobile elements

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Abstract. – We analyze the propagation of activity in a system of mobile automata. A number ρL^d of elements move as random walkers on a lattice of dimension d, while with a small probability p they can jump to any empty site in the system. We show that this system behaves as a Dynamic Small World (DSW) and present analytic and numerical results for several quantities. Our analysis shows that the persistence time T^* (equivalent to the persistence size L^* of small-world networks) scales as $T^* \sim (\rho p)^{-\tau}$, with $\tau = 1/(d+1)$.

The network of collaboration of actors in movies, of world airports, or of neural connections in the worm *Caenorhabditis elegans*, can be represented in the form of a graph. Persons, airports, and neurons stand for the nodes of the network, and a bond is drawn when two actors are cast together in a film, when a non-stop flight exists between two airports, or when a neural connection is present, respectively [1, 2]. The relational graphs resulting from these apparently disparate systems have two interesting properties that qualify them as *small-world* graphs: They are highly clustered at the local scale (similar to what happens in regular graphs), and the average number of bonds $\bar{\ell}$ that separate two randomly chosen nodes in a graph of size L scales as $\bar{\ell} \sim \log L$ (a characteristic property of random graphs).

Small-world (SW) graphs efficiently interpolate between regular and random graphs thanks to a small number pL^d of *shortcuts* (long-range connections, $p \ll 1$) which are superimposed on a regular lattice formed by L^d sites. Much attention has been devoted recently to the topological properties of such graphs, which determine the characteristic path length $\bar{\ell}$ and its scaling properties [3, 4], and to the effect that a small-world–like connectivity might have on the properties of dynamical systems [5, 6]. Up to now, the attention has been only focused on this relevant type of networks, SW. Nonetheless, the restriction of activity propagating *on a quenched, undirected graph*, where elements are fixed in their positions and information propagates in a symmetric way (from element A to element B as well as vice versa), is a strong one, and perhaps not always the best representation of a world formed by mobile elements

where connections (relational bonds) appear and disappear in the course of time. In this sense, recent approaches have tried to model the formation and evolution of the topology in SW and other networks through slow changes in their bonds [7, 8]. Our focus here is not in this large time scale, but on the shorter time scale in which the movement of the elements in the physical space is essential in defining contacts between nodes. As an example, consider the case of propagation of a tropical disease. Such events usually start in a "faraway" place (in the Euclidean sense), from which the carrier of the disease flies away. He or she arrives at a new destination and transmits the disease, and the process repeats with the new carriers. This is a clear example of a directed, dynamical graph, with moving elements which create and remove connections in a partly stochastic fashion. Indeed, in all cases where the carrier is not taking active part in the transmission of the signal, random encounters might play a main role when compared to the fixed network of acquaintances.

With the aim of analyzing the effect of movement in the transmission of a signal, we introduce a simple model for the formation of a Dynamic Small World (DSW), and study the propagation of activity among mobile elements in the system. We define quantities analogous to the characteristic path length $\bar{\ell}$ and to the clustering coefficient C in order to quantify the similarities with SW. Our analytical and numerical calculations show that DSW are quantitatively different from (quenched) SW.

The model is defined as follows. Consider a lattice formed by L^d sites embedded in a d-dimensional space, where ρL^d automata can move. The density of automata ρ is a parameter of the model. A time step consists of one move of all the active elements in the system plus the synchronous transmission of activity to new inactive neighbours. The precise implementation of these rules is as follows: i) Choose one of the active elements according to a random sequential updating (*i.e.* each active element will be updated once and only once in an uncorrelated fashion with respect to the previous move). With probability p, the automaton selects an empty site in the system and jumps there. With the complementary probability $1-p$, it looks for a free site in its neighbourhood and moves to it. The neighbourhood consists of the $2d$ adjacent sites. ii) Once all the active automata have moved, activity is simultaneously transmitted by each active element to any other inactive element in its neighbourhood. As initial condition an automaton is selected at random and activated at $t = 0$, the rest of the system remains inactive. The active element starts moving according to rule i). When an inactive element is met at a neighbouring site, activity is transmitted according to ii). We use periodic boundary conditions in our simulations.

The probability p of jumping to a faraway site is the equivalent of the density of shortcuts in SW. In that case, the activity would propagate to one new bond per time step, and one could define a velocity of propagation $v_{\mathrm{SW}} = 1$ [3]. In the DSW case, the time required to transmit the activity to the chosen element depends on the density of automata ρ, so we expect $v_{\mathrm{DSW}} = v(\rho) \leq 1$.

We start by defining two quantities analogous to the characteristic path length $\bar{\ell}(p)$ and the clustering coefficient $C(p)$ of SW (for details, see ref. [1]). In our case, the quantity equivalent to $\bar{\ell}$ is the characteristic time $T(\rho, p)$ for the activity starting at the seed to reach a randomly chosen element in the system, averaged over many independent realizations of the process. Because $v_{\mathrm{SW}} = 1$ in SW, $\bar{\ell}$ and T coincide. We will call the analogous of the clustering coefficient *neighbouring coefficient*, and define it in the following way. Consider the set of particles located in the neighbourhood of element i at time t, $\mathcal{N}_t(i)$. At the next time step, this set will be in principle different, since all active elements have tried to move to new sites. We define the neighbouring coefficient for element i at time t as $K_t(i; \rho, p) = (2d)^{-1}\mathcal{U}_t(i)$, where $\mathcal{U}_t(i) \equiv \mathrm{card}\{\mathcal{N}_t(i) \cap \mathcal{N}_{t-1}(i)\}$. The average neighbouring coefficient for a system with parameters p and ρ is $K(\rho, p) = \langle (\rho L^d)^{-1} \sum_i K_t(i; \rho, p) \rangle$, where the brackets $\langle . \rangle$ represent a

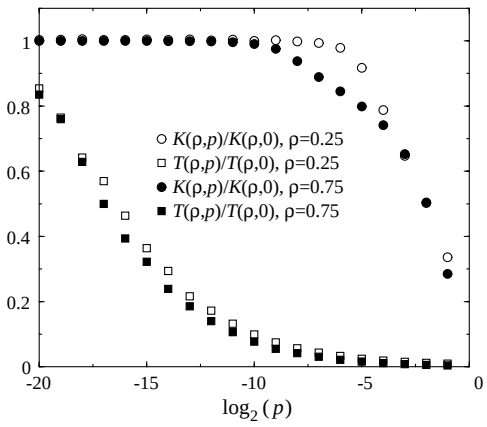

Fig. 1

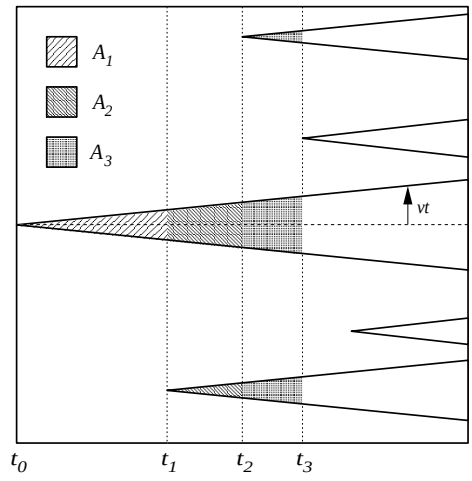

Fig. 2

Fig. 1 – Normalized characteristic time $T(\rho,p)/T(\rho,0)$ and normalized neighbouring coefficient $K(\rho,p)/K(\rho,0)$ as a function of p for two different values of the density, as shown in the legend, in a one-dimensional system of size $L = 5000$ (averages over 10^3 independent realizations have been performed).

Fig. 2 – Schematic representation of the propagation of activity in a Dynamic Small World. The initial condition corresponds to a single active particle which moves as a random walk. The spatio-temporal triangle signaling the spreading of activity has a half-angle $\alpha = \arctan v$. A second triangle starts the first time that an automaton jumps out of A_1, at a time $t_1 = (v\rho p)^{-1/2}$. When $A_k = A_1$, the k-th jump takes place.

temporal average. Our numerical simulations in $d = 1,\ 2$ reveal that the small-world behaviour of our system is well described through these two quantities (see fig. 1). Indeed, for a wide range of p values, the local configuration is maintained (the value of K is close to unity), while the typical number of time steps required to reach a randomly chosen automaton from the initial source is small and scales as $\log L$. This behaviour is characteristic of SW systems.

Our system admits a geometrical description suitable for calculating several quantities analytically. We will focus our interest on the growth of the volume $V(t)$ occupied by $a(t)$ active elements as a function of time t. There is a simple relation between the active volume and the number of active particles: $\rho V(t) = a(t)$. In the following, we restrict our calculations to the case $d = 1$. Some generalizations to arbitrary d are presented at the end of the letter.

At $t_0 = 0$, only one element is active. This element moves as a random walker during about $t \sim \rho^{-2}$ time steps (the average distance between elements is of order $1/\rho$), until it meets a second element at a neighbouring site, and activity is transmitted. This kind of spreading of activity persists for t_1 time steps, where t_1 signals the jump of the first active element to a domain disconnected (in terms of activity) from the previous one. For $p^{-1/2} \gg \rho^{-2}$ (contact through random walker moves is much more frequent than contact through long-distance jumps, see below), we can use a continuous approximation to the problem, as shown schematically in fig. 2. The vertical axis represents physical distance in a one-dimensional space, and the horizontal axis represents time. At time t, active elements cover a volume $V(t)$ equal to the sum of the lengths obtained from the intersection of a vertical line drawn at t with the triangles. For the particular case $p = 0$ the activity propagates at a constant velocity

$v(\rho)$ and, in the geometrical representation, only one triangle is present.

We can easily calculate the average time t_1 when the first jump (or shortcut) takes place. To this end, we make the annealed approximation $t \propto \langle\sqrt{n(t)}\rangle \sim \sqrt{\langle n(t)\rangle}$ (where $n(t)$ is the total number of attempted jumps up to time t), and obtain

$$t_1 = \sqrt{\frac{1}{v\rho p}} \,. \tag{1}$$

According to the picture in fig. 2, we can also calculate the times for the appearance of the next jumps,

$$t_2 = \frac{1+\sqrt{3}}{2} t_1 \,, \qquad t_3 = \frac{1}{2}\left[1 + \frac{\sqrt{3}}{3} + \frac{2\sqrt{6}}{3}\right] t_1 \,, \tag{2}$$

and so on. Note that jumps are not equidistant in time, and that they happen more and more frequently for longer times: $(t_1 - t_0) > (t_2 - t_1) > \ldots > (t_{k+1} - t_k) > \ldots$, and these differences go to zero for $k \to \infty$. In general, we can write an expression for the time by which k jumps have taken place,

$$t_1^2 k = \sum_{i=0}^{k-1} (t_k - t_i)^2 \,. \tag{3}$$

If we consider the new variable $\tau_k = (t_k - t_{k-1})/t_1$, the difference between consecutive times satisfies

$$k\tau_k^2 + 2\tau_k \sum_{i=1}^{k-1} i\tau_i = 1 \,. \tag{4}$$

We look for a solution of the form $\tau_k = (a + \epsilon_{k-1} - \epsilon_k)/k$. Direct substitution into eq. (4) yields $a = 1/\sqrt{2}$, $\epsilon_0 = a/2$, and $\epsilon_k \sim b'/k$, where b' is obtained by imposing a final condition for $k \to \infty$. In the original discrete variables, we obtain now

$$t_k = t_1 \sum_{i=1}^{k} \tau_i \simeq t_1 \left(\frac{1}{\sqrt{2}} \sum_{i=1}^{k} \frac{1}{i} + b - \frac{c}{k^2} \right) \,. \tag{5}$$

The coefficients of this expression are numerically obtained by fitting the iterative solution of eq. (4) to the expression above. This returns $b = 0.3124\ldots$, $c = 0.0337\ldots$.

For time $t < t_1$, the volume grows linearly, $V(t < t_1) \simeq 2vt$. We can also define the volume at the k-th jump (for $t_k \geq t_1$) as

$$V(k) = 2v \sum_{i=0}^{k-1} (t_k - t_i) = 2v \left(k t_k - \sum_{i=1}^{k-1} t_i \right) \,. \tag{6}$$

It follows from our previous calculations that

$$\frac{V(k)}{2vt_1} = \sum_{i=1}^{k} i\tau_i = \sum_{i=1}^{k} (a + \epsilon_{k-1} - \epsilon_k) = ak + \epsilon_0 - \epsilon_k \,. \tag{7}$$

Inverting eq. (5) and substituting above, we get the asymptotic expression

$$V(t) = \sqrt{2} v t_1 \exp\left[\sqrt{2}\left[\frac{t}{t_1} - \left(\frac{\boldsymbol{C}}{\sqrt{2}} + b\right)\right]\right] + \mathrm{O}(t^{-1}) \,, \tag{8}$$

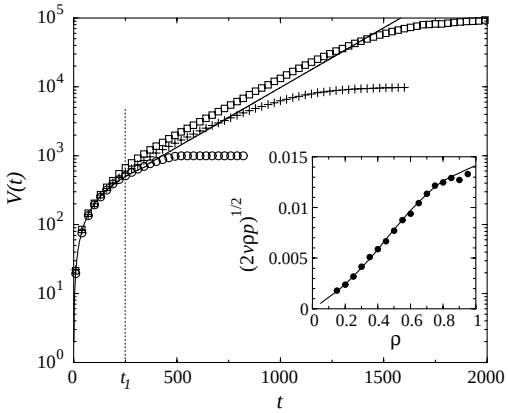

Fig. 3

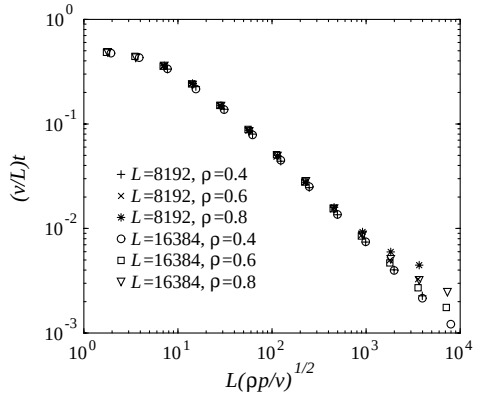

Fig. 4

Fig. 3 – Volume $V(t)$ covered by active particles. Main plot: The continuous line corresponds to $2vt$ for $t < t_1$ and to our solution, eq. (8), for $t > t_1$. These simulations were made for systems of size $L = 10^3$ (circles), 10^4 (crosses), and 10^5 (squares) with $\rho = 0.8$ and $p = 10^{-5}$. Inset: in the limit $t \gg t_1$ and $L \to \infty$, $V(t)$ grows exponentially at a rate $\sqrt{2v\rho p}$. This result (continuous line, with velocity numerically computed at $p = 0$) is compared with numerical interpolations for systems of size $L = 10^5$ (circles), with $\rho = 0.4$ and $p = 10^{-4}$ in the exponential regime.

Fig. 4 – Data collapse for the characteristic time T as a function of the rescaled system parameters. The simulations were performed in dimension $d = 1$ and averaged over 100 independent realizations. Other parameters are as shown in the legend. There are corrections to scaling for $p \to 1$, where the assumption $p^{-1/2} \gg \rho^{-2}$ breaks down.

where $\boldsymbol{C} = 0.577215\ldots$ is Euler's constant. Our numerical and analytical results are compared in fig. 3.

In these calculations, we have assumed that the time by which k jumps have taken place is a deterministic variable defined by $t_k \sim \sqrt{\langle n(t_k)\rangle}$, thus discarding fluctuations in $n(t)$. If we calculate the fluctuations in the jumping process, it turns out that their characteristic size is proportional to the approximate value t_k. The annealed approximation is quite good for ρ close to unity and $p \to 0$, but the effect of the fluctuations is usually visible in the fact that the apparent velocity at $t < t_1$ is larger than $v(\rho)$ when $p > 0$ (this explains the small discrepancy seen in fig. 3 for the largest size). The bending at finite $V(t)$ observed in fig. 3 is a finite-size effect: $V(t)$ saturates at the system size. The exponential regime is nonetheless well captured by our approach: In an infinite system, activity would propagate at a rate $\sqrt{2}/t_1 = \sqrt{2v\rho p}$, as shown in the inset of fig. 3.

In the light of the above results, we have studied the scaling properties of the characteristic time $T(\rho, p)$ with the system size L, the jumping probability p, and the density of automata ρ, in a way similar to what has been done for SW [9]. First, we need to estimate the dependence of a persistence time $T^* \equiv t_1$ [4] on the parameters of the system. Despite having a whole spectrum of characteristic times, as given by (5), all of the t_k scale in the same form as t_1. Hence, we conjecture that $T^* \propto (v\rho p)^{-1/2}$ in $d = 1$. Another relevant quantity with dimensions of time is $L/v(\rho)$. We propose the following scaling for T:

$$T \propto \frac{L}{v(\rho)} f\left(L\left(\frac{\rho p}{v(\rho)}\right)^{1/2}\right) , \tag{9}$$

where $f(z)$ is a function of the dimensionless parameter $z = L/L^*$, and we have defined the persistence size to be $L^* = vT^*$. We show in fig. 4 the resulting data collapse for several $T(\rho, p)$ curves, where $T(\rho, p)$ is the (minimum) time required for the activation of all particles in the system. Our numerical results support the scaling ansatz (9).

This result can be extended to dimensions $d > 1$. Our previous calculations rely on the determination of a persistence time T^*, which can be estimated in any dimension d from a geometrical representation of the system. In general, we can talk of activity propagating in a d-dimensional lattice, at a velocity $v(\rho)$, from a source placed at the center of a hypersphere of radius $r(t) = v(\rho)t$. The first jump in this d-dimensional space will take place when the volume of the $(d+1)$-dimensional cone (where time has been included as an additional dimension) contains $1/p$ automata. That is to say, the persistence time is defined by

$$1 = \frac{2\pi^{d/2}}{\Gamma(d/2)} v^d(\rho)\rho p \int_0^{T_d^*} t^d \mathrm{d}t \,, \tag{10}$$

which yields

$$T_d^* = \left[\frac{(d+1)\Gamma(d/2)}{2\pi^{d/2}} \frac{1}{v^d(\rho)\rho p} \right]^{1/(1+d)} . \tag{11}$$

It has been argued recently [4] that the presence of a finite persistence size in SW (or a finite persistence time in our case) is a finite-size effect implying that the transition that the system undergoes at $p \to 0$ is of first order. In both SW and DSW, $L^* \sim p^{-\tau}$, with an exponent $\tau_{\mathrm{SW}} = 1/d$ in the quenched case, while we have just shown that in the dynamical system $\tau_{\mathrm{DSW}} = 1/(d+1)$. We also expect a first-order critical transition for DSW at $p \to 0$. It is not difficult to understand where the difference in τ comes from. In DSW, effective jumps, that is, shortcuts able to carry the activity to an inactive domain, can originate in the bulk of the system, and not only at the boundaries of each active area, as is the case in SW [9]. The introduction of time as an additional "effective" dimension naturally leads to the calculated change in the scaling, and establishes a symmetry between time and space for DSW.

The efficiency of a system with shortcuts can be measured in terms of the average time that a signal takes to propagate, T^*. For fixed parameters L, p, and ρ, $T^*_{\mathrm{SW}} \simeq p^{-1}$, while $T^*_{\mathrm{DSW}} \simeq p^{-1/2}$. Since $p < 1$, $T^*_{\mathrm{DSW}} < T^*_{\mathrm{SW}}$. In other words, if we are going to design an algorithm to make a signal propagate, it would be better to invest effort in dynamical elements able to follow arbitrary paths than to construct fixed channels between fixed elements.

Dynamic Small Worlds might be an adequate description for a class of systems where the physical movement of the elements is the key ingredient for the transmission of a signal, and where a small-world–like behaviour may be present. Another example of this class of systems may be some species of ants, where two different types of individuals, characterized by two different speeds of movement, are known to coexist [10]. This moves our model to the field of collective computation, and suggests that a small-world behaviour might be relevant for the overall performance of such systems. Some authors have studied epidemics in models where elements are able to move on a lattice [11]. It would be interesting to analyze the dynamics of such a system on a DSW. According to the results presented here, they should differ quantitatively from current studies of epidemics on SW networks [5]. The case $p = 0$ of our model closely corresponds to an active process of the type $A + B \to 2B$ with strong inter-particle interactions. The spreading of the active front depends on the interplay between the movement of the active particle at the boundary and the position of the closest inactive particle in a non-trivial way. This problem is similar to that of a diffusing prey hunted by N predators [12].

Our model admits a number of generalizations. Recent studies have stressed the non-random structure of real networks. The scale-free topology of the World Wide Web (WWW), for instance, arises from a dynamical multiplicative process through which a small fraction of nodes accumulates most of the links [7]. In the DSW here presented all sites have equal probability of generating or receiving the next shortcut, resulting in a Poisson distribution of shortcuts per site. This rule can be easily modified if our sites are provided with memory, such that the probability of being selected by an automaton as a landing site depends on the number of automata which already visited it. This would generate a skewed distribution of shortcuts, with a small set of frequently visited sites. Hubs in the WWW would have their equivalent in "cities" (strongly preferred sites) in the physical space. Actually, the number of potential shortcuts of a real city (its population) distributes similarly to the number of connections per node in the WWW, and results from a similar growth process [13]. The presence of highly connected nodes in the WWW directly affects the tolerance of the network against errors [14]. It would be interesting to explore if the modification of the landing rule has also further consequences in the performance of DSW, for instance in the detection of the minimal path between two points by an automaton furnished with local information [15]. Further analysis of the DSW model here introduced will be the subject of forthcoming works.

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