Intermittency model for urban development

Susanna C. Manrubia* and Damián H. Zanette†
Fritz-Haber-Institut der Max-Planck-Gesellschaft, Faradayweg 4-6, 14195 Berlin, Germany

(Received 29 December 1997; revised manuscript received 18 March 1998)

The evolution of a stochastic reaction-diffusion model whose dynamics leads to the development of a strongly inhomogeneous, spatiotemporally intermittent density field is analytically and numerically studied. The processes underlying the model can be identified with those that govern urban development. The results for the reaction-diffusion model are thus compared with data obtained from real human demography. Statistical properties of urban distributions—in particular, the universal power law observed in the population frequency of cities—are successfully reproduced by the model. [S1063-651X(98)08107-0]

PACS number(s): 05.20.−y, 89.50.+r, 05.40.+j

I. INTRODUCTION: UNIVERSAL LAWS IN DEMOGRAPHIC DISTRIBUTION

The emergence of coherent macroscopic behavior is a distinctive feature in the dynamics of natural complex systems [1]. The interaction between their constituting elements originates cooperative evolution that can strongly differ from the individual dynamics. In the vast realm of biological phenomena, one of such macroscopic manifestations of complexity is social behavior, for instance, in the form of demographic evolution [2,3].

The macroscopic dynamics of complex systems is often characterized by the appearance of universal laws, whose validity does not depend on the value of the parameters that drive the microscopic evolution—which put in evidence the presence of general underlying mechanisms [4]. These universal laws are typically quantified in terms of characteristic exponents in scale-invariant distributions. A striking example of such universal laws occurs in the field of human demography. Some 50 years ago, it was already realized that urban distributions follow certain characteristic patterns that are repeatedly found in many countries, irrespectively of their social and economical conditions and history. The pioneering work of Zipf included the power-law distribution of city sizes according to their ''rank'' as one of the instances of what is nowadays known as Zipf law [2]. Those first studies by Zipf with a few well-known countries have been widely extended over the whole human population [3]. Such observations make it possible to conclude that a universal property to be ascribed to urban settlements is (1) the fraction $f(n)$ of cities with population n follows a power law $f(n)\propto n^{-r_o}$, with $r_o\approx 2$. (The subindex o stands here for ''observed'' quantities while, in the following, the results of our model will be denoted without subindex.) Remarkably, this property holds not only for cities, but also applies when rural population is included, at least within areas of even geographical conditions. Figure 1, for instance, shows the frequency $f(n)$ for the 2700 most populated cities of the world [5] and for the 2400 most populated cities of the United States of America [6]. The data for Switzerland, instead, stand for the largest 1300 municipalities [7], and that of the ten largest countries in south Europe corresponds to a total population counting in a square grid of 10 km^2 cells [5]. In these two cases—where $f(n)$ represents, respectively, the fraction of municipalities and grid cells with population n—rural population is also taken into account. It is then apparent that when population outside cities is also included, $f(n)\sim n^{-z_o}$ with $z_o\approx 2$. The distribution $f(n)\sim n^{-2}$ characterizes thus human settlements both inside and outside cities, and is extremely robust with respect to particular conditions of a certain country or region. In fact, the data for the world are expected to mainly reflect the situation of developing countries, the USA is an economically developed but relatively young country, whereas Switzerland and Germany are old countries with very stable populations but strongly dif-

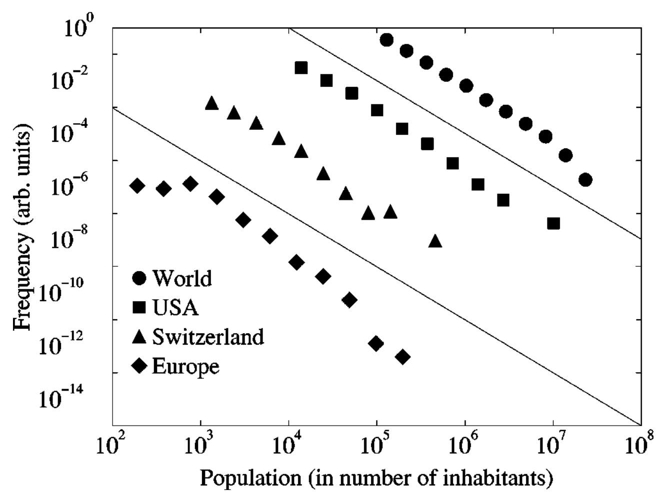

FIG. 1. Population frequency for the 2700 largest cities of the world [5] ($r_o=2.03\pm0.02$), for the 2400 largest cities of the United States of America [6] ($r_o=2.11\pm0.06$), for the 1300 largest municipalities of Switzerland [7] ($z_o=2.16\pm0.11$), and for the total population of the ten largest south European countries [5] ($z_o=2.17\pm0.18$) divided in equal areas of 10 km^2. For the sake of clarity, the data sets have been mutually shifted in the vertical direction. The straight lines have slope -2.

*On leave from Complex Systems Research Group, Universitat Politècnica de Catalunya, Campus Nord, B4, 08034 Barcelona, Spain.

†Permanent address: Consejo Nacional de Investigaciones Científicas y Técnicas, Centro Atómico Bariloche e Instituto Balseiro, 8400 S.C. de Bariloche, Río Negro, Argentina.

 © 1998 The American Physical Society

ferent recent history. Moreover, as already noticed by Zipf, the exponent $r_o \approx 2$ does not depend on time. This suggests that some process of population transport continuously restructures the relative sizes of cities, giving rise to a local growth in the population density that makes the power law persistent.

More recent studies on human demography have shown that the distribution of areas of satellite urban settlements around some large cities such as Berlin or London follow the same scaling law as for city populations [8]. Thus (2) the fraction $f(a)$ of cities with area a follows a power law $f(a) \propto a^{-s_o}$, with $s_o \approx 2$. Additional analysis has been focused on geometric properties of urban settlements. For instance, it has been pointed out that, within big cities, the amount of urbanized area decays exponentially from the compact core of maximum population density (often called ''central business district''). More precisely [8,9] (3) the probability $\rho(d)$ of finding an urbanized site at distance d from the compact city core decays as $\rho(d) \sim \exp(-\lambda_o d)$. The values of λ_o are found to vary within one order of magnitude. Moreover, the surfaces occupied by cities are delimited by rough boundaries, whose geometrical properties are also remarkably uniform [10,11]: (4) The fractal dimension D_o of a city boundary takes values between 1.2 and 1.4.

All these observations on real human demography seem to point out the likely existence of some fundamental, robust mechanisms underlying urban development. These mechanisms, in fact, have to be essentially independent of the specific social, economical, or political situation. In any other case, the above quoted functions would depend on parameters associated to such factors and thus lose their universality. It has been proposed [12] that this ubiquitous appearance of power-law distributions in urban systems might be the result of a self-organizing process that drives this system to a critical state—thus becoming independent of any detail that would describe specific current situations. Our approach to an explanation of the observed regularities is different. As we show in this paper, multiplicative stochastic processes [13] combined with a random diffusionlike transport mechanism are able to offer an explanation for the above described observations. This includes not only the quoted power-law distributions [properties (1) and (2)] but also the geometry of single cities [properties (3) and (4)].

In the following, we analyze in detail a previously introduced model of city formation [14]. The basic feature in its dynamics is the development of spatiotemporal intermittent patterns [15], where the relevant field accumulates in sharp spikes, that we associate with urban settlements. In this frame, urban development is part of a wide class of natural processes where intermittency plays a leading role. We mention, for instance, fully developed turbulence, where vorticity and energy dissipation concentrate along vortex lines [16]; magnetic fields in turbulent plasmas, that also become trapped by vortices [17]; autocatalytic chemical reactions [18]; and large-scale matter distribution in the Universe [19]. In Sec. II we describe the model and some possible variants, and give analytical results that explain a frequency $f(n) \sim n^{-2}$ in the resulting population [property (1)]. In Sec. III we test the robustness of the model under changes in the parameters and show that it reproduces quite satisfactorily all the properties of urban settlements quoted above. Finally, in Sec. IV we discuss our results and the interpretation of the model.

II. THE MODEL: TRANSPORT PROCESSES IN URBAN DEVELOPMENT

In our model, demographic structures are conceived as the result of two competing transport processes, which modify the distribution of population (or resources) in somewhat opposite ways. The first one represents the trend of human beings to concentrate in relatively small areas—the cities—to take advantage of the consequent concentration of resources in their social, technical, and economical activities. In fact, these activities are essentially based on the interchange of information between individuals inside the community, which is naturally enhanced if their mutual average distance shortens. *Cities attract people*, and their ''attraction power'' grows with their size. In the model, therefore, this process will be represented by multiplicative events that, at each time step and with a certain probability, increase the population of certain regions by a given factor. This growth is of course fed by population transport from other areas, where the population becomes consequently depleted. Thus the multiplicative events will occur in such a way that, although strong inhomogeneities develop, the total population is preserved in the average. As shown below—apart from this global, statistical conservation law—the multiplicative events are local in space, and can be compared with reactive processes in a reaction-diffusion model. This justifies the with $t<t'<t+1$, where $\xi(\mathbf{x},t)$ is a dichotomic stochastic process for each $\mathbf{x}$, defined as

$$\xi(\mathbf{x},t)=\begin{cases}(1-q)p^{-1}, & \text{with probability } p\\ q(1-p)^{-1}, & \text{with probability } 1-p.\end{cases} \tag{2}$$

The parameter q varies in principle within the interval $[0,1]$ but, due to the symmetry of the possible values of ξ under the change $(p,q)\to(1-p,1-q)$, it can be restricted to $[0,\min(1-p,1/2)]$.

The multiplicative process (1) is a generalization of the Zeldovich model for intermittency [20], which corresponds to $p=1/2$ and $q=0$. As advanced above, it preserves the average population,

$$\begin{aligned}\langle n(\mathbf{x},t')\rangle &= p\,\frac{1-q}{p}\langle n(\mathbf{x},t)\rangle+(1-p)\,\frac{q}{1-p}\langle n(\mathbf{x},t)\rangle\\ &=\langle n(\mathbf{x},t)\rangle,\end{aligned} \tag{3}$$

but it can be shown that, under the action of this sole process, higher population moments, $\langle n(\mathbf{x},t)^k\rangle=\Sigma_{\mathbf{x}} n(\mathbf{x},t)^k$ $(k>1)$, diverge as time elapses [18,21]. This divergence is in fact the mathematical characterization of intermittency and is a direct consequence of the appearance of strong inhomogeneities in the distribution $n(\mathbf{x},t)$. Sharp spikes appear where the events with probability p accumulate and the local population is at each step multiplied by $(1-q)p^{-1}>1$. In the remaining sites, whose number grows in time, the population decreases. Fluctuations thus play a key role in establishing the population distribution.

Without the action of diffusion, the reaction events (1)—as any purely multiplicative stochastic process—would give rise to a log-normal distribution for the population frequency $f(n)$. In fact, in Appendix A we show that the population frequency is in this case given by

$$f(n)=\frac{|A|n^{-1}}{\sqrt{\pi tp(1-p)}}\exp\left[-\frac{(A\ln n+Bt-pt)^2}{tp(1-p)}\right], \tag{4}$$

where A and B are constants depending on p and q only. It is well known that the log-normal distribution behaves as $f(n)\sim n^{-1}$ over a wide range that, in multiplicative stochastic evolution, increases as time elapses. In Eq. (4) the power-law approximation holds $n\ll\exp[|(B-p)/A|t]$, i.e., within a range that grows exponentially with time. Note that the remaining factor is still time dependent.

We conclude that a purely multiplicative stochastic process such as Eq. (1) is unable to produce the power-law exponent observed in real demographic distributions. Besides, in the intermediate region where an exponent—different from the observed one—is well defined, $f(n)\sim n^{-1}$, the population frequency is never stationary and depletes continuously to balance the growth in the extreme zones. As generated by this sole process, intermittency can qualitatively reproduce the strong heterogeneity of demographic distributions but fails to account for their detailed statistical properties. The combination with a diffusion mechanism, instead, will provide a successful explanation of those properties.

B. Diffusion processes

As discussed above, the second substep in the dynamics of our model corresponds to diffusion. In discrete-time evolution, diffusion can be performed by subtracting from each site $\mathbf{x}$ a certain fraction α $(0<\alpha<1)$ of the local population and homogeneously distributing that fraction on a prescribed neighborhood of $\mathbf{x}$. This can be expressed as

$$n(\mathbf{x},t+1)=(1-\alpha)n(\mathbf{x},t')+\frac{\alpha}{k}\sum_{\mathbf{x}'\in\{\mathbf{x}\}} n(\mathbf{x}',t'), \tag{5}$$

where the sum runs over the neighborhood $\{\mathbf{x}\}$ of $\mathbf{x}$ and k is the number of sites in $\{\mathbf{x}\}$.

Although this diffusion mechanism can be readily implemented in numerical simulations, the analytical problem—which can be put as a reaction-diffusion equation with a stochastic reaction term—proves to be rather complicated to solve in a two-dimensional space. In fact, to our knowledge, only some generic properties of its solution are known [18]. Thus we consider first a simplified version of the diffusion process, in which the neighborhood $\{\mathbf{x}\}$ of each site is extended to the whole system. As we show later, the effects of this form of ‘‘global’’ diffusion on the evolution given by the reaction process are essentially the same as those of local diffusion.

1. Global diffusion

When the neighborhood $\{\mathbf{x}\}$ where population from $\mathbf{x}$ spreads by diffusion is extended to the whole system, Eq. (5) becomes

$$n(\mathbf{x},t+1)=(1-\alpha)n(\mathbf{x},t')+\alpha n_0, \tag{6}$$

where n_0 is the (constant) average population per site, that we had initially fixed as $n_0=1$. The diffusion process becomes then effectively local in space and the whole effect of reaction and diffusion can be written in a single step as

$$n(\mathbf{x},t+1)=\begin{cases}(1-\alpha)(1-q)p^{-1}n(\mathbf{x},t)+\alpha, & \text{with probability } p\\ (1-\alpha)q(1-p)^{-1}n(\mathbf{x},t)+\alpha, & \text{with probability } 1-p.\end{cases} \tag{7}$$

This stochastic process, which reduces to Eq. (1) for $\alpha=0$, is not purely multiplicative. By virtue of its linearity, however, it can be exactly solved [21]. Due to the relatively strong effect of global diffusion, intermittency is here inhibited for sufficiently large values of α. For $q\to 0$, for instance, an inhomogeneous, peaked distribution develops for $\alpha<1-p$ only. Under these conditions, the population frequency $f(n)$—which is defined for $n_{\min}<n<n_{\max}(t)$, with $n_{\min}=\alpha$ and $n_{\max}(t)\sim[(1-\alpha)/p]^t$—reads

$$f(n)=\frac{1-\alpha-p}{p^2\alpha\ln[(1-\alpha)/p]}\times\left(1+\frac{1-\alpha-p}{p\alpha}n\right)^{-\ln p/\ln[p/(1-\alpha)]-1}, \qquad (8)$$

as shown in Appendix B. For large populations, it behaves as

$$f(n)\sim n^{-z}, \qquad (9)$$

with $z=1+\ln p/\ln[p/(1-\alpha)]$.

This result drives the attention to two noticeable facts. First, diffusion is able to modify the exponent in the power-law dependence of the population frequency determined by the reaction process in a nontrivial manner. The new exponent z is not universal, in the sense that it depends on the values of p and α. Note that for $\alpha\to 0$ and arbitrary p, $z\to 2$, which does not coincide with the exponent given by reactions ($z=1$) but agrees with the observed value. This implies that the limit of a purely reacting system ($\alpha=0$) is singular for this system, and that, for weak global diffusion, the model reproduces instead the exponent of real demography. The second point to be remarked as a difference with the purely reacting system is that, in the region where a power law in the population frequency is established by the competing effects of reaction and diffusion, a stationary distribution settles down. As we show in Sec. III, this is in full agreement with numerical results for the more realistic case of local diffusion.

2. Local diffusion

From previous results for the reaction-diffusion Zeldovich model [18], it is expected that, in contrast with global diffusion, local diffusion in low-dimensional systems is unable to inhibit the formation of strong inhomogeneities in the population distribution. According to simulations—and as suggested by our above discussion on global diffusion, where the resulting distribution does not depend on time—local diffusion is necessary to define a stationary population frequency in a growing range of values of n. Once this time-independent distribution is established, the main effect of diffusion consists of a population redistribution from the sites with higher values of n to low-populated sites.

In the intermediate stationary region, the profile of $f(n)$ can be fully ascribed to reactions. In the reaction substep (1), each value of n changes to $n'=(1-q)p^{-1}n$ with probability p or to $n'=q(1-p)^{-1}n$ with probability $1-p$. For an infinitely large system, these two contributions determine a new population frequency given by

$$f'(n')dn'=pf\left(\frac{p}{1-q}n'\right)d\left(\frac{p}{1-q}n'\right)+(1-p)f\left(\frac{1-p}{q}n'\right)d\left(\frac{1-p}{q}n'\right). \qquad (10)$$

Now, since the population frequency is supposed to be in a stationary state, we should have $f'\equiv f$, which produces the functional equation

$$f(n)=\frac{p^2}{1-q}f\left(\frac{p}{1-q}n\right)+\frac{(1-p)^2}{q}f\left(\frac{1-p}{q}n\right). \qquad (11)$$

For arbitrary values of p and q, its only two exact solutions with definite power-law dependence, $f(n)\propto n^{-z}$, are $f(n)=An^{-1}$ and $f(n)=An^{-2}$, with A a normalization constant. The exponent of the first one coincides with that of the purely reacting system. We recall, however, that in that case the population frequency was not stationary. The second solution is in full agreement with real demography.

The stability of both solutions with respect to the full dynamics of the model can be studied by means of numerical techniques. As shown in the next section, our simulations have always converged to a population frequency $f(n)\sim n^{-z}$ with $z\approx 2$, even from initially inhomogeneous distributions with large power-law regions where $z=1$. We cannot completely discard the possibility that some special initial distributions converge to $f(n)\sim n^{-1}$, but it is clear that a large class of initial conditions—including the homogeneous distribution we are interested in—evolve towards an asymptotic population frequency whose power-law decay is in excellent agreement with real data.

III. NUMERICAL RESULTS: POPULATION FREQUENCY AND CLUSTER ANALYSIS

In this section, we present the results of numerical simulations of the previous model. In our simulations, the population at each site of the $N\times N$-square lattice is a real number and local diffusion is defined as in Eq. (5).

Due to the finite size of the lattice, the numerical simulations are strongly affected by the fluctuations that drive the model. These lead the finite system to eventual extinction in a characteristic time of the order of N [22]. This effect is not representative of the dynamics on an infinite domain considered in the preceding section. We have solved this problem, on one hand, by working on rather large lattices ($N\approx 1000$) with periodic boundary conditions and, on the other, by adding a control process that avoids the spurious annihilation of population [23]. On large lattices ($N>500$)—where simulations are extremely time-consuming—the system is able to settle down in a transient (quasistationary) population distribution of the type derived in the preceding section, $f(n)\sim n^{-2}$. Eventually, however, the effect of fluctuations is felt by the finite system, the power-law distribution is destroyed, and the population is finally led to extinction. For smaller systems with periodic boundary conditions this effect of fluctuations can be avoided by adding extra population to already occupied sites when the total population falls below its initial value, as explained in the following.

In our simulations the initial population is N^2, with one unit per cell. At each time step, the total population is calculated. Suppose that at time t it has decreased down to $N^2 - \Delta_t$. In this situation, we divide the total difference Δ_t into N_t^o equal parts, N_t^o being the number of occupied sites at that time step. The population in each of these sites is then increased in an amount Δ_t / N_t^o. As a result, at time $t+1$ the total population has again been fixed to the initial one. In a typical realization ($N \sim 200$) this correction never exceeds 1% of the total population per time step, and becomes less important as N increases. On the other hand, the population is not corrected when it becomes greater than N^2, thus preserving the intermittent distribution over the lattice.

In [14] we have already presented numerical results for several parameter sets in which the parameter independence of the exponent $z \approx 2$ was shown. As a further verification of this robustness we have run simulations in which*T0 TD [*

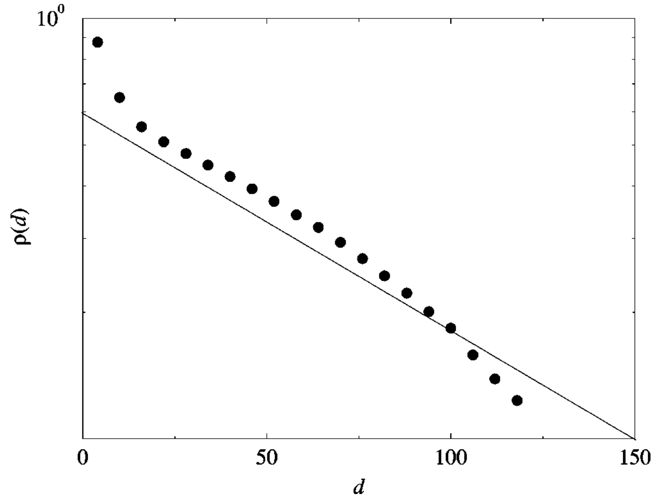

FIG. 3. Exponential decrease in the urbanization probability from the compact city core. The function is averaged over many independent realizations. The variable d represents the distance, in lattice units, to the most populated cell in every snapshot. In the vertical axis the probability of finding an occupied cell at distance d is represented.

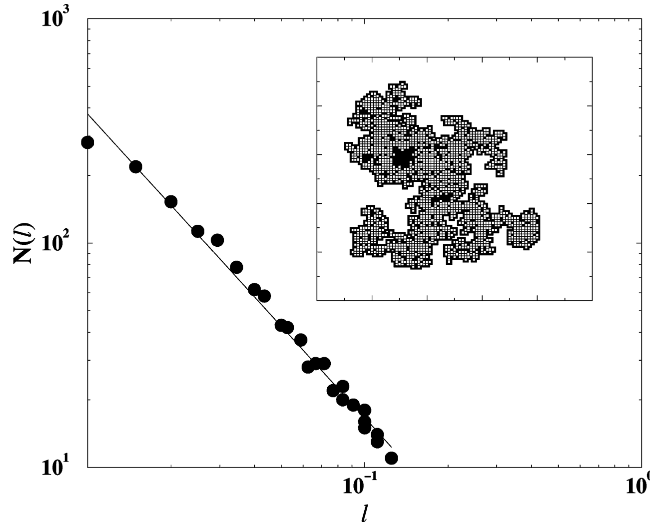

FIG. 4. Fractal dimension of the boundary of a typical large cluster. In the example displayed, the slope is $D \approx 1.3$. A systematic study of many different realizations of the model (for random parameters) returns values of the fractal dimension between 1.15 and 1.35, the higher the dimension the larger the cluster on the average.

counting algorithm [24]. A representative case of a cluster and its boundary, together with the obtained values for the fractal dimension, is displayed in Fig. 4. As already stated, the very nature of the model allows strong fluctuations in the system, and thus in the groups of connected cells. Our measures indicate that the fractal dimension of the largest cluster in the numerical simulations—which usually contains from 10^2 to 10^4 cells—varies between 1.15 and 1.35, depending basically on the size of the cluster, and is in good agreement with field measures [10].

IV. DISCUSSION AND CONCLUSION

We have studied in detail—by analytical and numerical means—a stochastic reaction-diffusion model whose dynamics is essentially driven by fluctuations and, as a consequence, gives rise to strongly intermittent patterns both in space and in time. The processes that govern its evolution—namely, diffusive transport and a kind of autocatalytic reaction—can be identified with the basic mechanisms underlying human urban development. We have thus compared the statistics of the spatiotemporal structures generated by the model with those observed in urban settlements and have found very good agreement in the exponents of the resulting power-law population frequency of cities. Other quantities, such as the distribution of city areas, the decay of population density, and the fractal dimension of city boundaries, have also been successfully compared. Our main quantitative results are summarized in Table I.

The exponent of the population frequency observed in real demography appears to be extremely uniform: it is independent of specific social, economical, or political (present or past) conditions. This suggest that a successful model for the processes that lead to this kind of universality has to be based on simple and rather general assumptions. We have analytically proven that our model generates the same exponent independently of the values of the parameters p, α, and q that define its evolution. These parameters characterize in fact particular conditions in the urbanization process. Large values of p and α should apply to regions with fast urban dynamics, such as in developing countries, whereas small values of those parameters describe slowly varying demography. In real situations, of course, the processes that govern urban evolution are expected to be modulated in time both in

TABLE I. Comparison between values for different exponents and functions obtained from observed data and from the analytical and theoretical results of the reaction-diffusion model presented here.

	Model	Observed data
Global demography	$z=-2$	$z_o=-2.0\pm 0.1$ [5]
City sizes (population)	$r=-1.90\pm 0.03$	$r_o=-2.03\pm 0.02$ [5]
City sizes (area)	$s=-1.93\pm 0.03$	$s_o=-1.80\pm 0.05$ [8]
Population-area law	$\beta=1.03\pm 0.03$	$\beta_o\approx 1$ [10,5]
Urbanized profile	Exponential	Exponential [8,9]
Dimension of the boundary	$1.15\leqslant D\leqslant 1.35$	$1.2\leqslant D_o\leqslant 1.4$ [10]

short scales—due to rapidly fluctuating factors—and in the long term—as the region under consideration develops socially and economically. Numerical simulations of our model have shown that even when p and α vary at random in time the exponent in the population frequency is preserved. The other quantities analyzed numerically—in particular, the distribution of city areas—have also been shown to exhibit well-defined power-law decays.

Note that the present model does not take into account the effects of birth-death events, which could be thought of as a severe limitation to the validity of our results. It can be argued, however, that—in real demography—global population growth becomes really important only when the population has settled down. Social stability is in fact a necessary condition for substantial growth. This has been the case, for instance, in European populations, where exponential growth began only when the nucleation of modern cities had already initiated [3]. Once a power-law population distribution has been established, exponential growth can only shift the distribution, but does not affect the relevant exponents.

There has recently been an increasing interest in proposing generic mechanisms able to explain the ubiquitous appearance of universal power-law distributions in physical, biological, social phenomena [4,12]. These mechanisms range from self-organizing evolution to purely stochastic multiplicative processes. In this paper we have shown, in connection with a specific problem in the area of global sociology, that models driven by stochastic processes can satisfactorily explain such universal laws in real systems.

ACKNOWLEDGMENTS

The authors are grateful to the Fritz Haber Institut for their warm hospitality in Berlin. They also wish to thank R. V. Solé, J. Delgado, A. S. Mikhailov, and P. Stange for useful discussions. S.C.M. acknowledges financial support from Max-Planck-Gesellschaft and from Grant No. DGYCIT PB94-1195. D.H.Z. acknowledges financial support from the Alexander von Humboldt Foundation, Germany, and from the Fundación Antorchas, Argentina.

APPENDIX A

When restricted to local reactions, the multiplicative stochastic process defined by Eqs. (1) and (2) reads

$$n(t+1)=\begin{cases}(1-q)p^{-1}n(t), & \text{with probability } p\\ q(1-p)^{-1}n(t), & \text{with probability } 1-p.\end{cases} \tag{A1}$$

This linear stochastic evolution equation is readily solved. For the initial condition $n(0)=1$, the possible values of $n(t)$ are

$$n_k(t)=\left(\frac{1-q}{p}\right)^k\left(\frac{q}{1-p}\right)^{t-k},$$

$$\text{with probability } p_k=C(t,k)p^k(1-p)^{t-k}, \tag{A2}$$

for $k=0,\dots,t$, where $C(t,k)=t!/k!(t-k)!$.

The index k acts here as a parameter labeling the values of $n(t)$. From Eq. (A2), it can be expressed as a function of n and t as

$$k=A\ \ln n+Bt, \tag{A3}$$

with $A=\{\ln[(1-q)(1-p)/qp]\}^{-1}$ and $B=A\ \ln[(1-p)/q]$. In addition, for large t and intermediate values of k, the probability p_k can be approximated by a Gaussian function:

$$p_k=\frac{1}{\sqrt{\pi tp(1-p)}}\exp\left[-\frac{(k-pt)^2}{tp(1-p)}\right]. \tag{A4}$$

Now, the probability distribution for n, $f(n)$, can be obtained from the relation $f(n)=p_k|\partial k/\partial n|$. Taking into account that $\partial k/\partial n=An^{-1}$, we get

$$f(n)=\frac{|A|n^{-1}}{\sqrt{\pi tp(1-p)}}\exp\left[-\frac{(A\ln n+Bt-pt)^2}{tp(1-p)}\right], \tag{A5}$$

namely, a log-normal distribution for n with a parametric dependence on time [cf. Eq. (4)]. As time elapses, the quadratic exponential in this distribution becomes broader and broader and, simultaneously, its maximum shifts at constant speed. For $|A\ \ln n|\ll|(B-p)t|$, i.e., for $n\ll\exp[|(B-p)/A|t]$, the exponential is practically constant as a function of n and the power-law approximation $f(n)\sim n^{-1}$ holds. The range where this approximation is valid grows thus exponentially with time.

APPENDIX B

For $q\to 0$, our reaction model with global diffusion, Eq. (7), becomes

$$n(t+1)=\begin{cases}An(t)+\alpha, & \text{with probability } p\\ \alpha, & \text{with probability } 1-p,\end{cases} \tag{B1}$$

with $A=(1-\alpha)/p>1$. The solution to this stochastic linear nonhomogeneous evolution equation gives, for the possible values of $n(t)$,

$$n_k(t)=\alpha\sum_{l=0}^{k}A^l=\alpha\frac{A^{k+1}-1}{A-1},$$

$$\text{with probability } p_k=(1-p)p^k, \tag{B2}$$

for $k=0,\dots,t-1$ and

$$n_{\max}=A^t+\alpha\frac{A^t-1}{A-1},\quad \text{with probability } p^t. \tag{B3}$$

Note that $n_{\max}>n_k$ for all k.

As in Appendix A, k plays here the role of a parameter labeling the values of n. In terms of n it reads

$$k=\frac{1}{\ln A}\ln\left(\frac{A-1}{\alpha}n+1\right)-1. \tag{B4}$$

Now, the probability distribution in terms of k is, for $n < n_{\max}$, $p_k = (1-p)p^k$, and does not depend on time. The distribution for n, $f(n) = p_k |dk/dn|$, reads then

$$f(n) = \frac{(A-1)(1-p)}{p\alpha \ln A}\left(\frac{A-1}{\alpha}n+1\right)^{\ln p/\ln A-1}, \quad \text{(B5)}$$

which coincides with Eq. (8) when A is replaced in terms of p and α.

This distribution is time independent, although the range where it holds, $n_{\min} = \alpha < n < n_{\max}$, grows exponentially as time elapses. In fact, $n_{\max} \sim A^t$ [cf. Eq. (B3)]. For large populations, $n \gg \alpha/(A-1)$, we have $f(n) \sim n^{-z}$ with $z = 1 - \ln p/\ln A > 0$.

[1] H. Haken, *Synergetics. An Introduction* (Springer, Berlin, 1978); A. S. Mikhailov, *Foundation of Synergetics I* (Springer, Berlin, 1990).

[2] G. K. Zipf, *Human Behavior and the Principle of Least Effort* (Addison-Wesley, Cambridge, MA, 1949).

[3] See, for instance, G. Haag, U. Müller, and K. G. Troitzch, *Economic Evolution and Demographic Change: Formal Models in Social Sciences* (Springer, New York, 1992); and A. Bailly, R. Ferras, and D. Pumain, *Encyclopédie de Géographie* (Economica, Paris, 1995).

[4] M. Schroeder, *Fractal, Chaos, Power Laws* (Freeman, New York, 1991).

[5] Source: National Center for Geographic Information and Analysis, Global Demography Project (http://ncgia.ucsb.edu/~uwe/pop.html#GLOBAL).

[6] Source: U.S. Census Bureau (http://www.census.gov/).

[7] Source: Swiss Federal Statistical Office (http://www.admin.ch/bfs/).

[8] H. A. Makse, S. Havlin, and H. E. Stanley, Nature (London) **377**, 608 (1995).

[9] C. Clark, J. R. Stat. Soc. A **114**, 490 (1951).

[10] M. Batty and P. Longley, *Fractal Cities* (Academic Press, San Diego, 1994).

[11] P. Frankhauser, *La Fractalité des Structures Urbaines* (Collection Villes, Anthropos, Paris, 1994).

[12] P. Bak, *How Nature Works* (Springer-Verlag, New York, 1996); P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987).

[13] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1992).

[14] D. H. Zanette and S. C. Manrubia, Phys. Rev. Lett. **79**, 523 (1997).

[15] Ya. B. Zeldovich, *The Almighty Chance* (World Scientific, Singapore, 1990).

[16] M. Tanaka and S. Kida, Phys. Fluids A **5**, 2079 (1993); S. Douady, Y. Couder, and M. Brachet, Phys. Rev. Lett. **67**, 983 (1991).

[17] Ya. B. Zeldovich, Zh. Éksp. Teor. Fiz. **32**, 552 (1957) [Sov. Phys. JETP **4**, 460 (1957)].

[18] A. S. Mikhailov, Physica A **188**, 367 (1992).

[19] S. F. Shandarin and Ya. B. Zeldovich, Rev. Mod. Phys. **61**, 185 (1989).

[20] Ya. B. Zeldovich, S. A. Molchanov, A. A. Ruzmaikin, and D.