Role of Intermittency in Urban Development: A Model of Large-Scale City Formation

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A stochastic model that incorporates the essential mechanisms supposed to govern city formation is numerically analyzed. The model generates intermittent spatiotemporal structures and predicts a power-law population distribution whose exponent is in excellent agreement with the universal exponent observed in real human demography. Preliminary results of cluster analysis of the model also coincide with actual data. We thus suggest that urban development at large scales could be driven by intermittency processes. [S0031-9007(97)03578-3]

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Social behavior is one of the most representative instances of complexity in natural systems. The interaction between their elements—for example, human beings—gives origin to cooperative evolution that strongly differs from the individual dynamics. At the macroscopic level, this phenomenon manifests itself in a wide variety of forms, such as demographic evolution, cultural and technical development, and economic activity [1].

A striking feature in the macroscopic dynamics of complex systems is the recurrent appearence of universal laws—quantified, typically, by characteristic exponents in scale-invariant distributions—that happen to be essentially independent of the details in the microscopic dynamics [2]. In physics, critical phenomena such as phase transitions provide a paradigmatic example of this universality. Several mechanisms have been proposed to explain in very general terms the occurrence of universality in complex behavior. Self-organization models [3,4], for instance, emphasize the role of nonlinear individual interactions in the formation of macroscopic spatiotemporal structures. This class of models has successfully explained the characteristic exponents of distributions occurring in very complex processes such as, for example, species macroevolution [5]. Stochastic fluctuations have also been pointed out as an important ingredient in the origin of universal laws [6].

The main aim of this Letter is to show that intermittency mechanisms [7] could play a relevant role in the universal properties of the (human) social phenomenon of city formation and global demographic development. We present a linear model whose dynamics is driven by a stochastic process, and show that it reproduces global statistical features of actual urban systems.

Spatiotemporal intermittency underlies the evolution of a wide class of real systems, ranging from population dynamics to turbulent fluids. In these systems, the relevant fields are different from zero in very localized regions of space and time. The whole evolution is therefore driven by the behavior of these small spots. In fully developed turbulence, for instance, the vorticity field and the rate of energy dissipation are concentrated along the vortex lines [8]. In plasmas, these lines trap also the magnetic field [9]. In population dynamics, fluctuations in birth and death rates produce strongly inhomogeneous distributions, characterized by sharp spikes [10]. These population peaks have their origin in the accumulation of favorable birth events and, since the main part of the population concentrates there, they dominate the evolution of the system in spite of their rarity. Intermittency has also been detected in the matter distribution of the Universe [11], and can have a relevant role in the kinetics of autocatalytic chemical reactions [12].

In qualitative terms, the demographic distribution of human beings on the Earth's surface strongly resembles intermittency patterns. Sharp peaks of concentrated population with a broad distribution of sizes—the cities—alternate with relatively large extensions where the population density is much lower. It has been shown that around huge urban centers, such as Berlin or London, the distribution of areas covered by satellite cities, towns, and villages follows a well-defined universal law [13]. The frequency of these relatively small population units as a function of their area A decays as A^{-r} with $r \approx 2$. This behavior, which is related to the occurrence of fractal structures in urban patterns [14], has been (partially) reproduced by means of correlated diffusion-limited aggregation models [13,15].

Most notably, the universal law for the size distribution of cities holds at much larger scales. This remarkable feature—which was formulated in semiquantitative terms by Zipf [16], some fifty years ago—has been recursively pointed out in subsequent research on demographic development [17]. Figure 1 shows a log-log plot of the frequency (in arbitrary units) as a function of the population n for the 2700 cities of the world with $n > 10^5$ inhabitants [18], the 2400 cities of the United States of America (U.S.A.) with $n > 10^4$ inhabitants [19], and the 1300 municipalities of Switzerland with $n > 10^3$ inhabitants [20]. It is clear that the frequency $f(n)$ shows a definite power-law dependence, $f(n) \sim n^{-r}$. From least-squares fitting of these data sets, we find $r = 2.03 \pm 0.05$ for the world (in the range of $n < 10^7$), $r = 2.1 \pm 0.1$ for the U.S.A., and $r = 2.0 \pm 0.1$ for Switzerland. The exponent r is therefore extremely uniform, $r \approx 2$, and the power-law

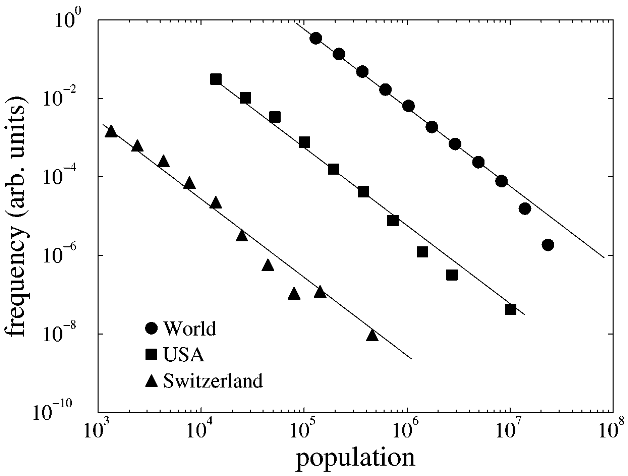

FIG. 1. Population distribution for the 2700 largest cities of the world, the 2400 largest cities of the United States of America (U.S.A.), and the 1300 largest municipalities of Switzerland. For the sake of clarity, the data sets have been mutually shifted in the vertical direction. The straight lines have slope -2.

dependence extends over several decades, in spite of the fact that the three data sets correspond to very different demographic, social, and economical conditions. In fact, the data for the world is expected to mainly reflect the situation of developing countries, the U.S.A. is an economically developed but young nation, whereas Switzerland is an old country with a relatively very stable population. In the following, we show that the power-law dependence in the frequency of city populations can be explained in terms of a very simple model based on the combination of stochastic reactionlike events and a diffusion process, which leads to the development of intermittency patterns. We are able to analytically explain the power-law exponent—which is independent of the model parameters—and numerically investigate other relevant distributions.

Consider a system evolving on a lattice at discrete time steps. The population in site x at time t, $n(x,t)$ is a real positive number. The system starts from a homogeneous distribution, for instance, $n(x,0) = 1$ for all x. The evolution consists of two time substeps. In the first one, the population at each site suffers a reactionlike stochastic process characterized by a probability p ($0 < p < 1$) and an additional parameter q ($0 \leq q \leq 1$):

$$n(x,t') = \begin{cases} (1-q)p^{-1}n(x,t), & \text{with probability } p\,, \\ q(1-p)^{-1}n(x,t), & \text{with probability } 1-p\,, \end{cases} \tag{1}$$

where $t < t' < t + 1$. Because of the symmetry of the two possible outputs for $n(x,t')$, the value of q can be restricted, for each value of p, to the interval $[0, 1/2]$. This linear random process is a generalization of the Zeldovich model for intermittency, which takes $p = 1/2$ and $q = 0$ [10]. Under the action of these reactionlike events, the population preserves—in a sufficiently large system—its mean value, but higher population moments, $\mu_k(t) = \sum_x n(x,t)^k$ ($k > 1$), diverge as time elapses. This divergence, which mathematically characterizes intermittency [21], is associated with the formation of strong inhomogeneities in the population. In fact, for small q ($q \ll 1 - p$), sharp spikes of increasing height appear where favorable events accumulate, whereas in the remaining sites—whose number grows in time—the population decreases rapidly. The dynamics is therefore dominated by fluctuations.

In connection with the formation of cities, Eq. (1) represents the progressive accumulation of population in the incipient urban centers. Indeed, since the average population is preserved, this mechanism can be interpreted as population transport from rural areas to a randomly chosen set of cities. As a consequence, each one of these cities grows—most plausibly—at a rate proportional to its size.

To complete the formulation of our model, the reaction process described by Eq. (1) is added with diffusion to nearest neighbors [22], which represents the natural spreading of population inside cities avoiding excessive local densities. Time-discrete diffusion is here characterized by a parameter α, which gives the fraction of population that abandons a given site at each step:

$$n(x, t+1) = (1-\alpha)n(x,t')\,. \tag{2}$$

The diffusing fraction $\alpha n(x,t')$ is uniformly distributed to the neighbors. For the Zeldovich model in continuous space and time, it has been shown that diffusion is unable to inhibit intermittency in low-dimensional systems [12]. We expect that the same result holds in our model, irrespectively of the values of p, q, and α.

In summary, at each time step all the sites are first submitted to the reactionlike process and then diffusion takes place. These two substeps are successively applied along the whole evolution. Figure 2 shows the population

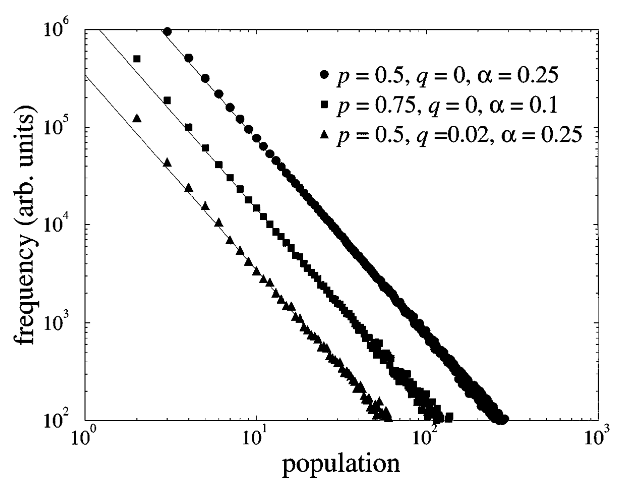

FIG. 2. Population distribution from numerical realizations of the present model on a 200×200 site lattice, for three parameter sets. The straight lines have slope -2.

frequency over the lattice sites averaged over time after a transient of 10^3 steps had elapsed, for three parameter sets. These numerical realizations were performed on a 200×200 site square lattice. The power-law dependence of the frequency $f(n)$ is extremely well defined, and the corresponding exponent is, for the three sets, $r = 2.01 \pm 0.01$, in complete agreement with the exponent of the data of Fig. 1. According to our simulations, this result is independent not only on the parameters p, q, and α but also on the lattice size, for moderately large sizes (greater than 50×50).

An exponent $r = 2$ in the population frequency $f(n)$ can be readily explained if we accept two main assumptions. In the first place, as numerical realizations show, $f(n)$ is a stationary distribution after a certain transient (whose length can depend on p, q, and α) has elapsed. The second assumption, suggested by the above quoted result on the Zeldovich model [12], is that diffusion is practically irrelevant in the evolution of $f(n)$. In fact, since diffusion would not be able to compete with intermittency, its effect should be qualitatively the same for any value of α, in particular, the same as for $\alpha \to 0$ (but $\alpha \neq 0$). The evolution of $f(n)$ can thus be thought of as follows, in terms of what happens for small α. In the reaction substep, each value of n changes to $n' = (1 - q)n/p$ with probability p or to $n' = qn/(1 - p)$ with probability $1 - p$. Therefore, the resulting frequency $f'(n')$ has two contributions:

$$f'(n')\,dn' = pf\left(\frac{p}{1-q}\,n'\right)d\left(\frac{p}{1-q}\,n'\right) + (1-p)f\left(\frac{1-p}{q}\,n'\right)d\left(\frac{1-p}{q}\,n'\right). \quad (3)$$

The main effect of diffusion consists, in the second substep, of a population redistribution from the sites with higher values of n' to low-populated sites, whereas n' remains practically unchanged in sites with moderate population. In $f(n)$, therefore, diffusion implies a depletion in the zone of high population and a consequent growth in the zone of small populations. In the intermediate region, $f(n)$ remains unchanged and we conclude that its evolution is essentially driven by reactions. Now, since the population frequency is supposed to be in a stationary state, we should have, in Eq. (3), $f' \equiv f$, which immediately implies

$$f(n) \propto n^{-2} \quad (4)$$

for intermediate values of n. In fact, if $f' = f$, $f(n) = An^{-2}$ is an *exact* solution to Eq. (3) for any value of A. Our result is also completely independent of the values of p, q, and α, although the limits of the region where it holds will in general depend on those parameters.

This analytical result, which fully explains the exponent observed for the population distribution in the numerical realizations of the present model (Fig. 2), is also in excellent agreement with real data (Fig. 1). The frequency $f(n)$ should however be considered as a rather rough characterization of the size distribution of cities. Indeed, urban centers should be identified, in our model, with clusters of connected sites whose population is above a certain threshold. This leads us to apply techniques of cluster analysis to the model, whose details and results will be presented in a forthcoming article [23]. In particular, the dynamics of formation and subsequent evolution of clusters is expected to depend on the reaction and diffusion parameters, although scaling properties can be still universal. Here, we present only some preliminary results on the distribution of areas and total populations of clusters. The area of a cluster is defined as the number of sites that it contains, and its population is the sum of $n(x,t)$ over those sites. As the system evolves, clusters vary in form and size—they can even aggregate, split, and spontaneously appear or disappear—but the distributions of areas and populations are, in temporal averages, well defined. The results presented here correspond to numerical realizations on a 200×200 site square lattice, averaged over time after transients have elapsed, with $p = 0.5$, $q = 0$, and $\alpha = 0.25$. As shown in Fig. 3, the frequency distributions of area and total population of clusters exhibit a power-law decay with an exponent which is again close to -2. For the area distribution, this exponent is in complete agreement with real data for large urban centers [13]. Note also that our results predict a linear correlation between area and total population inside clusters.

The coincidence between the universal exponent observed in real data of human population distributions and the exponent obtained from a model that incorporates only the essentials of plausible mechanisms in urban development is indeed remarkable. This coincidence seems to indicate that only a few, very elementary ingredients are really relevant in the dynamics of such a complex

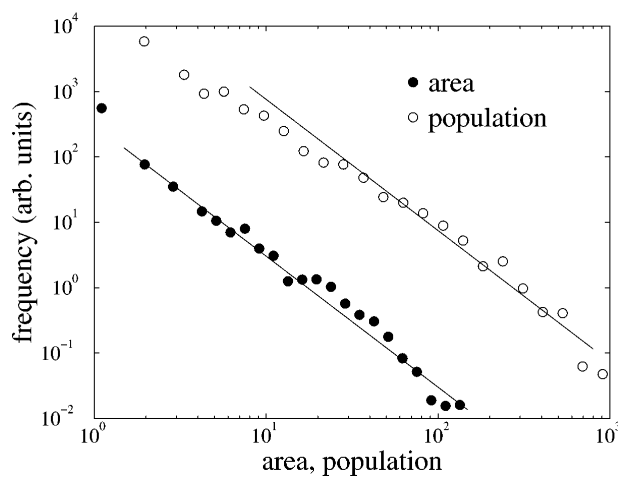

FIG. 3. Total-population and area distributions inside clusters of connected nonempty sites. These numerical results correspond to a 200×200 site lattice, with $p = 0.5$, $q = 0$, and $\alpha = 0.25$. For the sake of clarity, the data sets have been mutually shifted in the vertical direction. The straight lines have slope -2.

process. Although in principle our model is a reaction-diffusion system, the local dynamics is linear and the only form of "coupling" between elements—besides diffusion—comes from the average conservation of population in the reactionlike events. This coupling is however not a real interaction but a conservation law implicit in the involved stochastic process. Thus, the occurrence of universal complex behavior in this system should not be ascribed to a kind of self-organizing mechanism able to create (dissipative) spatiotemporal structures but rather to the combined and accumulative effects of random multiplicative events that give rise to intermittency.

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