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Distribution of repetitions of ancestors in genealogical trees

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Abstract

We calculate the probability distribution of repetitions of ancestors in a genealogical tree for simple neutral models of a closed population with sexual reproduction and non-overlapping generations. Each ancestor at generation g in the past has a weight w which is (up to a normalization) the number of times this ancestor appears in the genealogical tree of an individual at present. The distribution $P_g(w)$ of these weights reaches a stationary shape $P_\infty(w)$, for large g, i.e., for a large number of generations back in the past. For small w, $P_\infty(w)$ is a power law ($P_\infty(w) \sim w^\beta$), with a non-trivial exponent β which can be computed exactly using a standard procedure of the renormalization group approach. Some extensions of the model are discussed and the effect of these variants on the shape of $P_\infty(w)$ are analysed.

Keywords: Genealogy; Critical phenomena; Renormalization group

1. Introduction

Non-trivial power laws are known to characterize second-order phase transitions. A great success of the theory of critical phenomena has been to develop methods allowing to predict these power laws [1]. One of the most successful approaches used in the theory of critical phenomena is the renormalization group, which consists in trying to relate physical properties of a given system at different values of the external parameters (like the temperature or the magnetic field). In the last three or four decades, other non-trivial power laws [2] have been found in all kinds of systems: Transition to chaos

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by period doubling [3,4], geometrical problems like self-avoiding walks (which model polymers) and random walks [5], sand pile models and several other self-organized critical systems [6–8], coarsening [9], etc. In many cases, renormalization ideas could be extended to predict the exponents of these power laws.

In this work, we report recent results on simple models of genealogical trees [10]. When one looks at the distribution of repetitions in a genealogical tree (in the framework of the simple models defined below), one observes non-trivial power laws. The exponents of these power laws can be calculated *exactly* by writing a relation on the generating function of the weights of the ancestors (a quantity proportional to the number of times they appear in a genealogical tree) which has the form of a simple renormalization transformation. Beyond the intrinsic interest of these models to describe real genealogies, they constitute simple pedagogical examples for which renormalization ideas allow the exact prediction of non-trivial exponents.

2. Neutral models of genealogical trees

2.1. The random parent model

Let us first consider a simple neutral model of a closed population with sexual reproduction. By definition of the model, the population size at generation g in the past is N_g and each individual at generation g has two parents chosen at random among the N_{g+1} individuals in the previous generation $g+1$. Here g counts the number of past generations and so increases as one climbs up a genealogical tree. For simplicity we will consider either a population of constant size ($N_g = N$) or a population size increasing exponentially with an average number $p/2$ of offsprings per couple, i.e., $N_g = (2/p)^g N_0$ as g counts the number of past generations; N_0 is the size of the population at present, while the constant size case corresponds to $p = 2$.

A related model was introduced to study the genetic similarity between individuals in a population evolving under sexual reproduction [11], although there the two parents were distinct. We do not exclude this case here.

Clearly, the number of branches of the genealogical tree of any individual increases like 2^g and, as soon as the number of branches exceeds N_g, there should be repetitions in this tree. Let us denote by $r_i^{(\alpha)}(g)$ the number of times that an individual i living at generation g in the past appears in the genealogical tree of individual α. At generation $g = 0$, the only individual in the tree of α is α itself, therefore

$$r_i^{(\alpha)}(0) = \delta_{i,\alpha} \tag{1}$$

and the evolution of these repetitions satisfies the recursion

$$r_i^{(\alpha)}(g+1) = \sum_{j \text{ children of } i} r_j^{(\alpha)}(g) \,. \tag{2}$$

The quantity we want to consider is the probability $H(r,g)$ that an individual living at generation g in the past appears r times in the genealogical tree of individual α (living at generation 0). Normalization implies

$$\sum_{r\geqslant 0} H(r,g) = 1 , \tag{3}$$

the initial condition (1) gives

$$H(r,0) = \frac{1}{N_0}\delta_{r,1} + \left(1 - \frac{1}{N_0}\right)\delta_{r,0} , \tag{4}$$

and the fact that each individual has two parents at the previous generation gives

$$\sum_{r\geqslant 0} rH(r,g) = \frac{2^g}{N_g} . \tag{5}$$

These probabilities $H(r,g)$ can be measured by simulating small systems through a Monte Carlo procedure: For each individual of a population at generation g, two parents are chosen at random among the N_{g+1} individuals at generation $g+1$. Fig. 1 shows the results of such simulations for two populations of constant sizes, $N_g = N_0$, for several values of g with $N_0 = 1000$ in Fig. 1a and $N_0 = 10\,000$ in Fig. 1b.

We see that for small g there are very few repetitions and $H(r,g)$ decreases very fast with r. On the other hand, when g increases, the shape of $H(r,g)$ becomes independent of g and of the population size N, with a clear power law at small r and a fast decay at large r. Fig. 2 shows the distribution $H(r,g)$ for several values of g and a population which increases exponentially with time, $N_g = 3^{10-g}2^g$. Here, again, the shape becomes stationary when g is large enough but N_g is still large. This stationary shape is different from the one seen in Fig. 1. The shape of $H(r,g)$ becomes stationary for large N_g and large g in the sense that one gets a fixed distribution by an appropriate rescaling. In fact, introducing the rescaled quantities w and $P_g(w)$

$$w = \frac{N_g}{2^g} r , \tag{6}$$

$$P_g(w) = \frac{2^g}{N_g} H(r,g) , \tag{7}$$

where w can be considered as a continuous variable for $N_g \ll 2^g$, (3) and (5) transform into

$$\int P_g(w)\,\mathrm{d}w = \int wP_g(w)\,\mathrm{d}w = 1 , \tag{8}$$

and we expect $P_g(w)$ to become a fixed distribution $P_\infty(w)$. This means that if we associate to each individual i in the tree of α at generation g in the past a weight defined by

$$w_i^{(\alpha)} = \frac{N_g}{2^g}\, r_i^{(\alpha)} \tag{9}$$

the distribution of these weights becomes stationary in the scaling limit.

From (2) and (9) it is clear that these weights satisfy

$$w_i^{(\alpha)}(g+1) = \frac{N_{g+1}}{2N_g} \sum_{j \text{ children of } i} w_j^{(\alpha)}(g) . \tag{10}$$

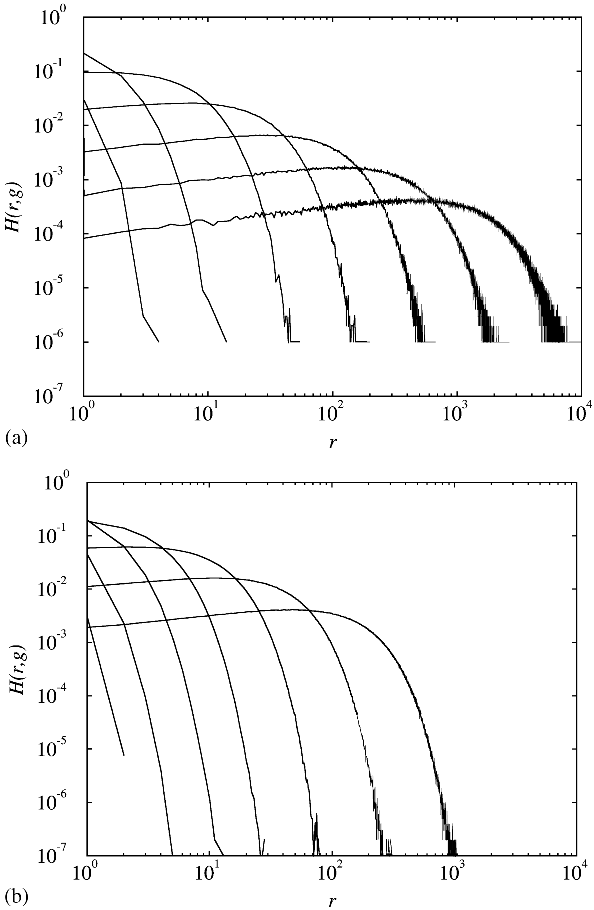

Fig. 1. Probability distribution $H(r,g)$ of r repetitions after g generations ($H(0,g)$ is not shown) at $g=5,9,12,14,16,18$, and 20 for a population of constant size. In Fig. 1a, $N=1000$ and in Fig. 1b, $N=10\,000$. Both figures show averages over 1000 samples.

As we limit ourselves to the case of a population increasing exponentially at rate $p/2$ per generation (so that $N_g=(2/p)^g N_0$), (10) reduces to

$$w_i^{(\alpha)}(g+1)=\frac{1}{p}\sum_{j \text{ children of } i} w_j^{(\alpha)}(g)\,. \tag{11}$$

The ratio $w_i^{(\alpha)}(g)/N_g$ can be interpreted as the probability of reaching individual i by randomly climbing up the genealogical tree of α. In the particular case of a population of constant size ($p=2$), the factor $\frac{1}{2}$ in (11) is easy to understand. For a population of increasing size ($p>2$), there is a factor $1/p$ in (11) instead of $\frac{1}{2}$ because of the factor N_g in the definition (9) of the weights $w_i^{(\alpha)}$.

The key observation which allows one to calculate the distribution $P_g(w)$ in the scaling limit (large g and large N_g) is that, for large N_g and for large g, *the random*

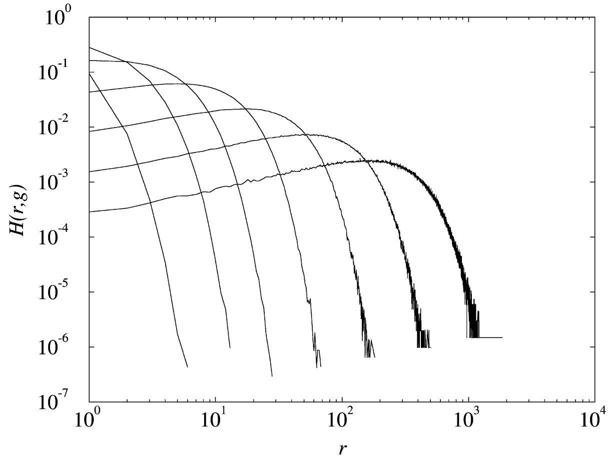

Fig. 2. Probability distribution $H(r,g)$ for a population size increasing by a factor $\frac{3}{2}$ at each generation. Here $N_g = 3^{10-g}2^g$, and averages over 5000 samples are performed. The generations shown are $g = 8, 10, 11, 12, 13, 14$, and 15.

variables $w_j^{(\alpha)}$ *which appear in the r.h.s. of* (11) *become independent*. This is due to the fact that (at least in the model we consider) the weights $w_j^{(\alpha)}(g)$ (of brothers and sisters) in the r.h.s. of (11) are uncorrelated. This independence, which is discussed in the appendix, will be the basis of the calculation of the fixed distribution $P_\infty(w)$ in the following sections.

2.2. Variants of the model

One can consider some variants of the model defined above, for instance:

- At each generation one could form fixed couples by making random pairs and assign to each individual at generation g one of these pairs (of parents) chosen at random at the previous generation $(g+1)$. In this case the correlations between the weights w_g would again be small in the scaling limit and they can be ignored in the r.h.s. of (11).
- One can also consider an imaginary situation where each individual has $p' \neq 2$ parents (instead of $p' = 2$). In this case, the definition of the weights (9) should be replaced by

$$w_i^{(\alpha)} = \frac{N_g}{(p')^g}\, r_i^{(\alpha)} \tag{12}$$

to keep $P_g(w)$ normalized as in (8). For a population of constant size $N_g = N$, the evolution of the weights (11) becomes

$$w_i^{(\alpha)}(g+1) = \frac{1}{p'} \sum_{j \text{ children of } i} w_j^{(\alpha)}(g)\,. \tag{13}$$

As shown in the appendix, in the scaling limit, the correlations on the r.h.s. of (13) can be neglected in this case too.

In the remaining of this work, we try to predict the stationary shape $P_\infty(w)$.

3. Generating function

The fact that the weights in the r.h.s. of (11) are uncorrelated greatly simplifies the problem. One can then consider that $w_i^{(\alpha)}(g+1)$ is the sum of k independent identically distributed random variables $w_j^{(\alpha)}(g)$, where k is itself random. The probability q_k of k is clearly

$$q_k = \binom{2N_g}{k} \left(\frac{1}{N_{g+1}}\right)^k \left(1 - \frac{1}{N_{g+1}}\right)^{2N_g - k} ,$$

which, for large N_g, becomes (using the fact that $N_{g+1} = 2N_g/p$) a Poisson distribution

$$q_k = \frac{p^k}{k!} \, \mathrm{e}^{-p} \, . \tag{14}$$

Therefore, for large N_g, the number k of terms (k is the number of children of i) in the r.h.s. of (11) is randomly distributed according to (14) and these k terms are uncorrelated. This becomes a problem of branching processes [9]. If one introduces the generating function $Q(\lambda, g)$

$$Q(\lambda, g) = \langle \exp[\lambda \; w_i^{(\alpha)}(g)] \rangle \tag{15}$$

and uses (11) and the fact that the weights are independent, one finds that $Q(\lambda, g)$ satisfies

$$Q(\lambda, g+1) = \sum_{k \geqslant 0} q_k \; Q\left(\frac{\lambda}{p}, g\right)^k = \exp\left[-p + p\, Q\left(\frac{\lambda}{p}, g\right)\right] . \tag{16}$$

The normalization (9) of the $w_i^{(\alpha)}(g)$ implies that we have for all g

$$Q(0, g) = Q'(0, g) = 1 \, . \tag{17}$$

Recursions similar to (16) appear in the theory of branching processes, in particular in the Galton–Watson process, already introduced in the 19th century to study the problem of the extinction of families [12].

From (15) and (16), one can easily obtain recursions for the moments of the weigths $w_i^{(\alpha)}$,

$$\langle w(g+1) \rangle = \langle w(g) \rangle = 1 \, , \tag{18}$$

$$\langle w^2(g+1) \rangle = \frac{1}{p} \langle w^2(g) \rangle + 1 \, , \tag{19}$$

$$\langle w^3(g+1) \rangle = \frac{1}{p^2} \langle w^3(g) \rangle + \frac{3}{p} \langle w^2(g) \rangle + 1 \, , \tag{20}$$

$$\langle w^4(g+1)\rangle = \frac{1}{p^3}\langle w^4(g)\rangle + \frac{4}{p^2}\langle w^3(g)\rangle + \frac{3}{p^2}\langle w^2(g)\rangle^2 + \frac{6}{p}\langle w^2(g)\rangle + 1 \tag{21}$$

and so on. We see that for large g, each moment of $w_i^{(\alpha)}(g)$ has a limiting value, as expected from the observation in the previous section that $P_g(w)$ converges to a fixed distribution $P_\infty(w)$ such that

$$Q(\lambda,\infty) = \int_0^\infty e^{\lambda w}\, P_\infty(w)\,\mathrm{d}w\,. \tag{22}$$

The limiting values of these moments

$$\langle w^2(\infty)\rangle = \frac{p}{(p-1)}\,, \tag{23}$$

$$\langle w^3(\infty)\rangle = \frac{p^2(p+2)}{(p-1)(p^2-1)}\,, \tag{24}$$

$$\langle w^4(\infty)\rangle = \frac{p^3(p^3+5p^2+6p+6)}{(p-1)(p^2-1)(p^3-1)}\,, \tag{25}$$

etc., can be obtained directly by expanding the solution $Q(\lambda,\infty)$ of

$$Q(\lambda,\infty) = \exp\left[-p + pQ\left(\frac{\lambda}{p},\infty\right)\right] \tag{26}$$

around $\lambda = 0$ (choosing as normalization $Q'(\lambda,\infty) = 1$),

$$Q(\lambda,\infty) = 1 + \lambda + \frac{p}{2(p-1)}\lambda^2 + \frac{p^2(p+2)}{6(p-1)(p^2-1)}\lambda^3 + \frac{p^3(p^3+5p^2+6p+6)}{24(p-1)(p^2-1)(p^3-1)}\lambda^4 + O(\lambda^5)\,. \tag{27}$$

Several other properties of $Q(\lambda,\infty)$ can be obtained from the fixed point equation (26) or from the recursion (16). The simplest one is the limit

$$S = \lim_{\lambda\to-\infty} Q(\lambda,\infty)\,, \tag{28}$$

where S is the solution ($S \neq 1$) of

$$S = e^{-p+pS}\,. \tag{29}$$

This limiting value ($S = 0.20318787\ldots$ for a population of constant size, i.e., $p = 2$) is the coefficient of $\delta(w)$ in $P_\infty(w)$ and so is the fraction of the population whose descendants become extinct: There is a fraction e^{-p} of the population with no children, a fraction $e^{-p+pe^{-p}} - e^{-p}$ of the population with children but no grandchildren, and so on, and the sum of all these contributions gives S.

Eqs. (16) and (26) have the form of a real-space renormalization [13]. As a consequence, one can predict that for $\lambda \to -\infty$, $Q(\lambda,\infty)$ approaches its limit as a power law,

$$Q(\lambda,\infty) - S \sim |\lambda|^{-\beta-1}\,, \tag{30}$$

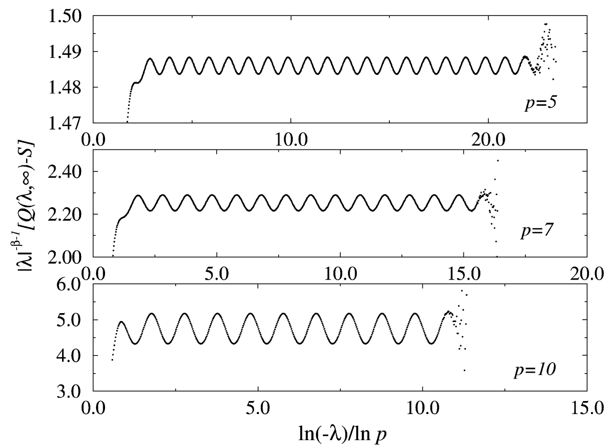

Fig. 3. The product $|\lambda|^{-\beta-1}[Q(\lambda,\infty)-S]$ versus $\ln(-\lambda)/\ln p$ for $p=5,7,10$. We see clearly the periodic nature of the amplitude predicted by (33). Discrepancies at small $-\lambda$ are due to the fact that the asymptotic regime is not yet reached. At too large $-\lambda$, rounding errors in the difference $Q(\lambda,\infty)-S$ make the result noisy and unreliable.

where the exponent β must be

$$\beta = -2 - \frac{\ln S}{\ln p} \tag{31}$$

for the terms of order $|\lambda|^{-\beta-1}$ on both sides of (26) to be equal. For $p=2$, this gives $\beta = 0.2991138\ldots$ and (22) implies that at small w, the distribution $P_\infty(w)$ is a power law

$$P_\infty(w) \sim w^\beta \tag{32}$$

with β given by (31), in agreement with the results of the simulations shown in Figs. 1 and 2.

In fact, for $\lambda \to -\infty$, the leading contribution in the difference $Q(\lambda,\infty)-S$ consistent with (26) is

$$Q(\lambda,\infty) - S \simeq |\lambda|^{-\beta-1} F_p\left(\frac{\ln \lambda}{\ln p}\right), \tag{33}$$

where $F_p(z)$ is an arbitrary periodic function (not necessarily constant) of period 1 (i.e., $F_p(z+1) = F_p(z)$). Such periodic amplitudes are often present in the critical behaviour of systems which have a discrete scale invariance [14–16]. It is easy to calculate numerically the function $Q(\lambda,\infty)$ for all values of λ from the fixed point equation (26) which relates λ to points λ/p^n arbitrarily close to 0, where the linear approximation $Q(\lambda,\infty) \simeq 1+\lambda = O(\lambda)^2$ becomes excellent. Using this procedure, we could determine (Fig. 3) the combination $[Q(\lambda,\infty)-S]|\lambda|^{-\beta-1}$ and the non-constant periodic nature of the amplitude $F_p(z)$ is visible if p is large enough. The analytic determination of $F_p(z)$ is in principle possible [17,18] for p close to 1, but remains difficult for arbitrary p.

The knowledge of the periodic function $F_p(z)$ determines in principle the whole expansion of $Q(\lambda,\infty)$ in the limit $\lambda \to -\infty$. If we look for a solution of (26) which starts as (33) as $\lambda \to -\infty$, one finds by equating the two sides of (26) order by order in powers of $|\lambda|^{-\beta-1}$,

$$Q(\lambda,\infty) = S + \frac{F_p(\ln\lambda/\ln p)}{|\lambda|^{\beta+1}} + \frac{p}{2(pS-1)}\left[\frac{F_p(\ln\lambda/\ln p)}{|\lambda|^{\beta+1}}\right]^2 + \frac{p^2(pS+2)}{6(pS-1)((pS)^2-1)}\left[\frac{F_p(\ln\lambda/\ln p)}{|\lambda|^{\beta+1}}\right]^3 + \cdots . \tag{34}$$

In addition to the moments (23)–(25) of $P_\infty(w)$ (which are given by the expansion (27) of $Q(\lambda,\infty)$) and the exact values (29) and (31) of S and β, let us just mention two properties of the solution of (26) which we checked by rather complicated ways, and that we prefer to leave as conjectures:

- $Q(\lambda,\infty)$ is analytic in the whole complex plane of λ
- $Q(\lambda,\infty)$ grows extremely fast (faster than the exponential of the exponential ... of the exponential of λ) as $\lambda \to \infty$. As a consequence, for large w, $P_\infty(w)$ decays faster than any exponential but slower than any stretched exponential (of exponent larger than 1) and even

$$1 \ll \frac{-\ln P_\infty(w)}{w} \ll \ln w . \tag{35}$$

All the discussions of the present section can be repeated in the case of having p' parents. If we limit ourselves to a population of constant size (as we did to obtain (13)), we find that $Q(\lambda,\infty)$ satisfies the same fixed point equation (26) as above with p replaced by p'

$$Q(\lambda,\infty) = \exp\left[-p' + p'Q\left(\frac{\lambda}{p'},\infty\right)\right] . \tag{36}$$

This means that the distribution of the weights w is exactly the same for the cases of (i) two parents and a population size increasing exponentially by a factor $p/2$ at each generation and (ii) a population of constant size with p parents per individual. This can be checked by comparing Figs. 2 and 4, where we show the distributions $H(r,g)$ for a population of constant size $N=1000$ and $N=10\,000$ with three parents per individual.

4. Perturbation theories

Despite its simplicity, it is not easy to extract more information on the function $Q(\lambda,\infty)$ and consequently on the distribution $P_\infty(w)$ from the fixed point equation (26). There are, however, two limiting cases around which one can apply a perturbation theory and extract a few more properties of the fixed distribution: p close to 1 and p very large.

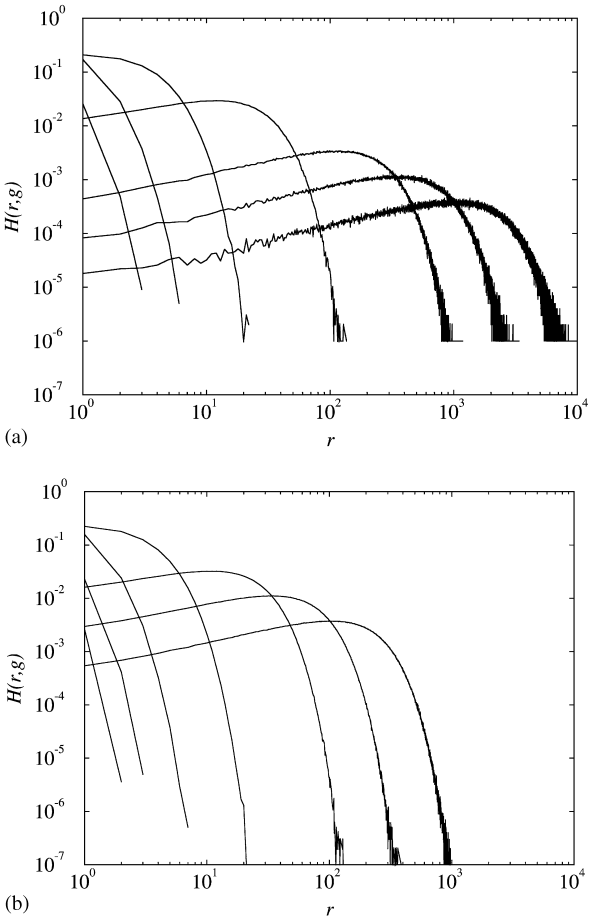

Fig. 4. (a) The function $H(r,g)$ for a population of constant size with $N=1000$ and (b) $N=10\,000$ when the number p of parents is 3. The generations shown are $g=3,5,7,9,11,12$, and 13.

4.1. p close to 1

One can see from (23)–(25) that when $p\to 1$, the successive moments of the weight w diverge like $\langle w^n\rangle \sim (p-1)^{1-n}$. This indicates that if one writes

$$p=1+\varepsilon \tag{37}$$

the solution of the fixed point equation (26) can be expanded in the following way:

$$Q(\lambda,\infty)=1+\varepsilon f_1\left(\frac{\lambda}{\varepsilon}\right)+\varepsilon^2 f_2\left(\frac{\lambda}{\varepsilon}\right)+\varepsilon^3 f_3\left(\frac{\lambda}{\varepsilon}\right)+\varepsilon^4 f_4\left(\frac{\lambda}{\varepsilon}\right)+\cdots, \tag{38}$$

where the functions $f_1, f_2, \ldots$ resume the most divergent terms in the perturbative expansion (27) in the range $\lambda = O(\varepsilon)$. If we insert expansion (38) into (26) we get, by equating the two sides order by order in ε, a hierarchy of differential equations for the functions $f_1, f_2, \ldots$ which can be solved and lead to

$$f_1(y) = \frac{y}{1 - y/2}\,, \tag{39}$$

$$f_2(y) = \frac{2}{3}\frac{y^2}{(1-y/2)^2} + \frac{1}{3}\frac{y}{(1-y/2)^2}\ln\left[1 - \frac{y}{2}\right]\,, \tag{40}$$

$$f_3(y) = \frac{14y^3 - 3y^2}{36(1-y/2)^3} + \frac{17y^2 - 6y}{36(1-y/2)^3}\ln\left[1 - \frac{y}{2}\right] + \frac{y^2 + 2y}{36(1-y/2)^3}\ln^2\left[1 - \frac{y}{2}\right]\,. \tag{41}$$

Comparing these expressions for large negative y with (34), one gets the expansions of S, β

$$S = 1 - 2\varepsilon + \frac{8}{3}\varepsilon^2 - \frac{28}{9}\varepsilon^3 + O(\varepsilon^4)\,,$$

$$\beta = \frac{\varepsilon}{3} - \frac{\varepsilon^2}{18} + \frac{19}{540}\varepsilon^3 + O(\varepsilon^4)\,,$$

which both agree with what one would get by directly expanding (29) and (31). What the small ε expansion gives us in addition is the function $F_p(z)$ which is found to be a constant function of z to all orders in powers of ε,

$$F_p(z) = 4\varepsilon^2 - \frac{32}{3}\varepsilon^3 + 18\varepsilon^4 + O(\varepsilon^5)\,.$$

The non-constant nature of $F_p(z)$ does not show up in the expansion in powers of ε. It is a non-perturbative contribution (which vanishes to all orders in $\varepsilon = p - 1$) which could be calculated [17] using WKB-like techniques [18].

From (38) to (40) and definition (22) one finds that, for small ε, the continuous part of $P_\infty(w)$ is an exponential

$$P_\infty(w) \simeq \left(1 - 2\varepsilon + \frac{8\varepsilon^2}{3}\right)\delta(w) + 4\varepsilon^2 e^{-2\varepsilon w}\,.$$

Corrections to this exponential shape are extractable from higher order terms (f_2, $f_3, \ldots$).

4.2. Large p

The other case which can be dealt with perturbatively is the limit of large p. If p is large and $\lambda = O(p^{1/2})$, the solution of (26) is given by

$$\ln Q(\lambda, \infty) = \lambda + \frac{\lambda^2}{2p} + \frac{\lambda^3}{6p^2} + \left[\frac{\lambda^2}{2p^2} + \frac{\lambda^4}{24p^3}\right] + \left[\frac{\lambda^3}{2p^3} + \frac{\lambda^5}{120p^4}\right] + O(p^{-2})\,, \tag{42}$$

where each term represents a new order in powers of $p^{-1/2}$. This implies that $P_\infty(w)$ can be written in terms of $x = w - 1$ in the range $x \sim p^{-1/2}$ as

$$P_\infty(w) \simeq \sqrt{\frac{p}{2\pi}} e^{-px^2/2}\left[1 + \left(\frac{px^3}{6} - \frac{x}{2}\right) + \left(\frac{p^2x^6}{72} - \frac{px^4}{6} + \frac{7x^2}{8} - \frac{7}{12p}\right)\right.$$

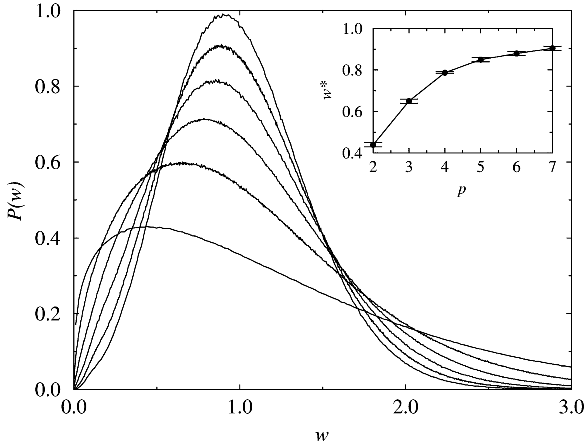

Fig. 5. The fixed distribution $P_\infty(w)$ (the delta function contribution at $w=0$ is not shown) for $p=2$, with $N=2^{15}$ and $g=25$; $p=3$, $N=3^{10}$, $g=18$; $p=4$, $N=4^8$, $g=14$; $p=5$, $N=5^6$, $g=11$; $p=6$, $N=6^6$, $g=11$; and $p=7$, $N=7^5$, $g=9$. Averages over 1000 realizations have been carried out. The inset shows how the maximum w^* varies with p.

$$+\left(\frac{p^3x^9}{1296}-\frac{p^2x^7}{48}+\frac{19px^5}{80}-\frac{95x^3}{144}-\frac{x}{8p}\right)+\cdots\Bigg], \tag{43}$$

where each parenthesis represents a new order in $p^{-1/2}$. The Gaussian shape in (43) is not a surprise considering that, for large p, each weight becomes the sum of a large number of independent contributions.

One property which can be extracted from (43) is the location of the maximum w^* of $P_\infty(w)$

$$w^* = 1-\frac{1}{2p}-\frac{25}{24p^2}+O\left(\frac{1}{p^3}\right) . \tag{44}$$

Fig. 5 shows the shapes (obtained by random samplings populations of constant sizes with p parents per individual) of the distribution $P_\infty(w)$ for several choices of p. The inset shows the values of w^* extracted from these data. They agree with the prediction (44) that the maximum approaches 1 with corrections of order $1/p$ as p becomes large.

5. Conclusions

We have seen that for simple neutral models of evolution with random mating, the distribution of ancestors repetitions in the genealogical tree of a present individual becomes stationary, with a fixed shape $P_\infty(w)$ which can be described by a fixed point equation of the type (26). This shape is the same if one considers a population increasing exponentially at rate $p/2$ per generation with two parents per individual or a population of constant size with p parents per individual.

The fixed point equation (26) allows one to determine exactly the exponent β which characterizes $P_\infty(w)$ at small w. The determination of β from (26) is very reminiscent of the way one finds exponents in the renormalization group approach of critical phenomena. Other properties (large w behavior, amplitude of the power law, ...) of the fixed distribution $P_\infty(w)$ are in principle extractable from (26) but are more difficult to obtain than the exponent β.

The present work admits several extensions. In particular, one may consider the case where the probabilities q_k (that an individual has k children) is arbitrary (instead of Poissonian as in (14)). The fixed point equation (26) becomes then simply

$$Q(\lambda,\infty) = \sum_k q_k Q\left(\frac{\lambda}{p},\infty\right)^k$$

and starting from this new fixed point equation, one can essentially repeat all the above calculations, including the determination of the exponent β. If all the q_k vanish for $k > k_{\max}$, one can see that for large λ,

$$\ln Q(\lambda,\infty) \sim \lambda^{\ln k_{\max}/\ln p} .$$

Consequently, the distribution $P_\infty(w)$ becomes a stretched exponential for large w,

$$\ln P_\infty(w) \sim -w^{\ln k_{\max}/\ln(k_{\max}/p)} .$$

Recursions similar to (11) describe the distribution of constraints in granular media [19]. In such cases, the number of grains in direct contact and supporting the weight of a given grain is variable. This would correspond to considering that the number p' of parents is no longer constant over the whole population but may vary from individual to individual.

Finally, let us mention that an interesting aspect of the problem is the calculation of the correlations between the genealogies of several contemporary individuals. One can show [20] that for large g, the weights of all the ancestors of two distinct individuals in the same population become the same after a number of generations $g_c \propto \ln N$.

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Appendix A. The correlations of the weights

In this appendix we show, by calculating moments of the weights $w_j^{(\alpha)}(g)$, that correlations become negligible in the r.h.s. of (11) and (13).

A.1. The case of a varying population size with two parents per individual

It is convenient to rewrite (11) as

$$w_i^{(\alpha)}(g+1) = \frac{1}{p}\sum_{j=1}^{N_g} \chi(i,j)\, w_j^{(\alpha)}(g)\,, \tag{A.1}$$

where

$$\chi(i,j) = \begin{cases} 0 & \text{if } i \text{ is not a parent of } j \\ 1 & \text{if } i \text{ is one of the two parents of } j \\ 2 & \text{if } i \text{ is the two parents of } j\,. \end{cases} \tag{A.2}$$

For the random parent model of Section 2 (where each parent of j is chosen at random among all the individuals of the previous generation), $\chi(i,j)=0$ with probability $(1-1/N_{g+1})^2$, $\chi(i,j)=1$ with probability $2(1-1/N_{g+1})/N_{g+1}$ and $\chi(i,j)=2$ with probability $1/N_{g+1}^2$ (as we did not exclude choosing the same parent twice). Moreover, there is no correlation between $\chi(i,j)$ and $\chi(i',j')$ if $j \neq j'$. Lastly, $\chi(i,j)$ and $\chi(i',j)$ are correlated for $i \neq i'$ and

$$\langle \chi(i,j)\chi(i',j)\rangle = \frac{2}{N_{g+1}^2}\,. \tag{A.3}$$

This correlation together with

$$\langle \chi(i,j)\rangle = \frac{2}{N_{g+1}}\,, \tag{A.4}$$

$$\langle \chi(i,j)^2\rangle = \frac{2}{N_{g+1}} + \frac{2}{N_{g+1}^2}\,, \tag{A.5}$$

$$\langle \chi(i,j)\chi(i',j')\rangle = \frac{4}{N_{g+1}^2} \quad \text{for } j \neq j' \tag{A.6}$$

when used in (A.1) leads to

$$\langle w_i(g+1)\rangle = \langle w_i(g)\rangle$$

as expected, since the definition (6) of w was chosen to keep $\langle w\rangle = 1$, and

$$\langle w_i(g+1)^2\rangle = \left(\frac{1}{p} + \frac{1}{pN_{g+1}}\right)\langle w_i(g)^2\rangle + \left(1 - \frac{2}{pN_{g+1}}\right)\langle w_i(g)w_{i'}(g)\rangle\,, \tag{A.7}$$

$$\langle w_i(g+1)w_{i'}(g+1)\rangle = \frac{1}{pN_{g+1}}\langle w_i(g)^2\rangle + \left(1 - \frac{2}{pN_{g+1}}\right)\langle w_i(g)w_{i'}(g)\rangle\,, \tag{A.8}$$

where $i \neq i'$ (the index $^{(\alpha)}$ has been omitted for simplicity).

From (1), (2) and (6) we know that $\sum_i w_i(g) = N_g$, and $\langle w_i(g)\rangle = 1$. Thus, for $i \neq i'$

$$\langle w_i(g)w_{i'}(g)\rangle = \frac{N_g - \langle w_i(g)^2\rangle}{N_g - 1} \tag{A.9}$$

and (A.7) becomes

$$\langle w_i(g+1)^2\rangle = \left(\frac{1}{p} - \frac{1}{pN_{g+1}} - \frac{1}{N_g-1} + \frac{2}{pN_{g+1}(N_g-1)}\right)\langle w_i(g)^2\rangle + \left(1-\frac{2}{pN_{g+1}}\right)\frac{N_g}{N_g-1}. \tag{A.10}$$

So far this evolution equation is exact.

If we consider that all the N_g's are very large (A.10) becomes

$$\langle w_i(g+1)^2\rangle = \frac{1}{p}\langle w_i(g)^2\rangle + 1 \tag{A.11}$$

so that for large g (in fact, g should not be too large to keep N_g large enough, more precisely g should be such that $(p/2)^g \ll N_0 \ll p^g$), the second moment of w has a limiting value $\langle w_i(g)^2\rangle \to p/(p-1)$ and we see from (A.9) that

$$\langle w_i(g)w_{i'}(g)\rangle \to 1 = \langle w\rangle^2. \tag{A.12}$$

When one repeats the above calculation for higher correlations (we did it up to three-point correlations), one finds that the correlations between the terms in the r.h.s. of (A.1) are negligible. This indicates that these correlations can be neglected (of course a complete proof that all correlations are negligible in the scaling limit would be much better than our guess based on the computation of the lowest correlations).

One can repeat the above calculation of correlations for several variants of the model, like those discussed at the end of Section 2. The exact formulae (A.7), (A.8) and (A.10) are modified but one always finds that, in the scaling regime, they reduce to (A.11) and (A.12), meaning that the correlations could be ignored.

A.2. The case of a population of constant size with p' parents per individual

Let us consider only the case where each individual has p' parents. To keep the notations simple, we will limit the calculation to the case of a population of constant size

$$N_g = N\,.$$

One can then follow the same steps as above. Starting from (13), one replaces (A.1) by

$$w_i^{(\alpha)}(g+1) = \frac{1}{p'}\sum_{j=1}^{N_g}\chi(i,j)w_j^{(\alpha)}(g). \tag{A.13}$$

Correlations (A.3)–(A.6) become in this case

$$\langle\chi(i,j)\chi(i',j)\rangle = \frac{p'(p'-1)}{N^2} \quad \text{for } i \neq i'\,, \tag{A.14}$$

$$\langle\chi(i,j)\rangle = \frac{p'}{N}\,, \tag{A.15}$$

$$\langle \chi(i,j)^2 \rangle = \frac{p'}{N} + \frac{p'(p'-1)}{N^2}\,, \tag{A.16}$$

$$\langle \chi(i,j)\chi(i',j') \rangle = \frac{p'^2}{N^2} \quad \text{for } j \neq j' \tag{A.17}$$

and (A.7) and (A.8) read

$$\langle w_i(g+1)^2 \rangle = \left(\frac{1}{p'} + \frac{p'-1}{p'N}\right) \langle w_i(g)^2 \rangle + \left(1 - \frac{1}{N}\right) \langle w_i(g) w_{i'}(g) \rangle\,, \tag{A.18}$$

$$\langle w_i(g+1) w_{i'}(g+1) \rangle = \frac{p'-1}{p'N} \langle w_i(g)^2 \rangle + \left(1 - \frac{1}{N}\right) \langle w_i(g) w_{i'}(g) \rangle\,. \tag{A.19}$$

For large g and large N, we see (using the fact that $\sum_i w_i(g) = N$) that $\langle w_i(g)^2 \rangle \to p'/(p'-1)$ and $\langle w_i(g) w_{i'}(g) \rangle \to 1$ as $g \to \infty$. This again indicates that correlations can be neglected for large g and large N.

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